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MATHEMATICS 2024. Vol. 34. Issue 2. Pp. 222–247.

MSC2020: 34D20, 34D10, 34D05

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ON THE STABILITY IN VARIATION OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS WITH PERTURBATIONS

In this paper, we investigate the problem of stability in variation of solutions for nonautonomous differential equations. Some new sufficient conditions for the asymptotic or exponential stability for some classes of nonlinear time-varying differential equations are presented by using Lyapunov functions that are not necessarily smooth. The proposed approach for stability analysis is based on the determination of the bounds that characterize the asymptotic convergence of the solutions to a certain closed set containing the origin. Furthermore, some illustrative examples are given to prove the validity of the main results.

Keywords: nonautonomous differential equations, perturbation, Lyapunov functions, asymptotic stability.

DOI: 10.35634/vm240204

§ 1. Introduction

Asymptotic stability is a fundamental concept in the qualitative theory of dynamical systems, and it plays a significant role in numerous applications of the theory in almost all domains where dynamic effects are present. The general theory of the stability of motion is presented in monographs [19, 29–31, 36, 37]. The use of Lyapunov functions is important in the study of the stability of solutions of systems of differential equations. This approach is based on the finding that asymptotic stability is closely related to the existence of a Lyapunov function, which is a proper, nonnegative function that decreases along the system's paths that do not evolve in the invariant set and vanishes only on an invariant set. There are different methods for the stability analysis via Lyapunov functions for time-varying differential equations [3–8, 33–35]. However, for nonlinear systems of differential equations the construction of such functions is a complex problem. It turns out that the Lyapunov functions can be used not only for studying the stability of solutions but also the asymptotic behaviors analysis [2, 12, 38, 39].

It is well known that the second method of Lyapunov [29, 30] provides sufficient conditions to ensure various types of stability for a dynamical system described by ordinary differential equations with perturbations [20, 21, 23–28]. The perturbation term could result from errors in modeling a nonlinear system, aging of parameters or uncertainties and disturbances. In general, we know some information on the upper bound of the term of perturbation. Given two solutions to a dynamical system with initial conditions that are close at the same value of time, these solutions will remain close over the entire time interval and not just at the initial time. This motivates us to study the problem of uniform asymptotic stability of perturbed systems by assuming that the nominal associated system is globally uniformly asymptotic stable "in variation" (see [10] and the references therein) under some restrictions on the size of perturbations. The notion of uniform Lipschitz stability [17, 18] lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation and uniform stability in variation on the other side. The problem concerning the asymptotical convergence of solutions of differential equations by using continuously differentiable function with a negative definite derivative is a classical one. In certain cases, Gronwall inequalities and Lyapunov theorems provide stability conditions that do not require knowledge of the system trajectories. However, for some dynamical systems one takes a Lyapunov function which is not necessarily differentiable that arises naturally.

The stability and asymptotic behavior of nonlinear systems have been studied in [10] using the analogue of Alekseev's variation of constants formula for nonlinear systems (see [1]). Dannan and Elaydi have introduced the uniform Lipschitz stability [17], for systems of differential equations. It was shown in [17] that, for linear systems, the notions of uniform Lipschitz stability and that of uniform stability are equivalent. However, for nonlinear systems, the two notions are quite distinct [17]. In fact, uniform Lipschitz stability lies somewhere between uniform stability on one side and the notions of asymptotic stability in variation developped by Brauer [10] and uniform stability in variation of Brauer and Strauss [9] on the other side.

In this paper, we use Lyapunov's second method to define the behavior of solutions of some nonlinear differential equations. Especially, we are looking for the asymptotic stability of the solutions using Lyapunov function which are not necessarily smooth. We prove that the solutions are bounded and converge to a certain closed set containing the origin. Furthermore, some examples are given to show the applicability of the main results.

§ 2. Stability analysis

The qualitative behavior of the solutions of perturbed nonlinear systems of differential equations is often studied by obtaining a Lyapunov function for the unperturbed system and using it as a Lyapunov function for the perturbed system. We wish to investigate the properties of solutions of a system of differential equations when a Lyapunov function is known whose derivative, in the sense of Dini, along solutions of the system satisfies a negative definitive condition under the presence of perturbations. Let us consider a time-varying system described by the following time-varying differential equation:

$$\dot{x} = f(t, x) \tag{2.1}$$

where $f\colon \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and locally Lipschitz with respect to x such that $f(t, 0) = 0,\ \forall t \geqslant 0$, and the associated perturbed systems:

$$\dot{x} = f(t, x) + g(t) \tag{2.2}$$

where $t \in \mathbb{R}^+$, $g\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ is a continuous function.

In the sequel, we give some of the main definitions that we need to study the asymptotic behavior of the solutions. The notion of stability will be given in the sense of *"Stability in variation"* (see [10, 11]).

Consider the time-varying system (2.1). Unless otherwise stated, we assume throughout the paper that the function $f(.,.)$ encountered is sufficiently smooth. We often omit arguments of function to simplify notation; $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$; $B_r = \{x \in \mathbb{R}^n \mid \|x\| \leqslant r\}$, $r > 0$; $\|A\| = \max_{\|x\|=1} \|Ax\|$ is the norm of a matrix A.

For any $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$, we will denote by $x(t, t_0, x_0)$, or simply by $x(t)$, the unique solution of (2.1) at time t_0 starting from the point x_0. We have, $\forall t \geqslant t_0 \geqslant 0$,

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f\big(s, x(s, t_0, x_0)\big)\, ds.$$

Let $f_x(t, x)$ be the matrix whose element in the ith row, jth column is the partial derivative of the ith component of f with respect to the jth component of x, i. e., $f_x = \left\{\dfrac{\partial f_i}{\partial x_j}\right\}_{i,j=1}^{n}$. Let $x(t, t_0, x_0)$ be the solution of (2.1). We have

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0}\Big(x(t, t_0, x_0)\Big)$$

is a solution of the variational system:

$$\dot{z} = f_x\big(t, x(t, t_0, x_0)\big)z. \tag{2.3}$$

The matrix $\Phi(t, t_0, x_0)$ is called *the fundamental matrix solution of* (2.3) *with respect to the solution* $x(t, t_0, x_0)$. We have $\Phi(t, t_0, x_0)\Big|_{t=t_0} = I$.

We will define the notion of stability in terms of variational system with respect the solution $x = 0$ (see [9, 10, 17, 18]).

Definition 2.1. The solution $x = 0$ of (2.1) is said to be *globally uniformly Lipschitz stable in variation* [17, Definition 1.4], or, in other words, *globally uniformly stable in variation* [9],[2, Definition 1.3] if there exists a positive constant M such that

$$\|\Phi(t, t_0, x_0)\| \leqslant M \quad \text{for all } \ t \geqslant t_0 \geqslant 0 \ \text{ and } \ x_0 \in \mathbb{R}^n.$$

Note that, the solution $x(t, t_0, x_0)$ of (2.1) satisfies the following equality:

$$x(t, t_0, x_0) = \left(\int_0^1 \Phi(t, t_0, sx_0)\, ds\right)x_0. \tag{2.4}$$

From (2.4), it follows that, if the solution $x = 0$ of (2.1) is globally uniformly stable in variation, then

$$\|x(t, t_0, x_0)\| \leqslant M\|x_0\| \quad \text{for all } \ t \geqslant t_0 \geqslant 0 \ \text{ and } \ x_0 \in \mathbb{R}^n.$$

For linear systems, global uniform stability in variation and uniform stability coincide [17, Theorem 2.1]. The following simple property can be stated for the variational system.

Proposition 2.1. *If the zero solution of* (2.1) *is globally uniformly stable in variation, then the zero solution is globally uniformly stable for*

$$\dot{z} = f_x\big(t, x(t, t_0, 0)\big)z. \tag{2.5}$$

P r o o f. Indeed, let zero be globally uniformly stable in variation for (2.1). Then there exists $M > 0$ such that $\forall\, t \geqslant t_0 \geqslant 0 \;\forall\, x_0 \in \mathbb{R}^n$ $\|\Phi(t, t_0, x_0)\| \leqslant M$. By (2.4), $\forall\, t \geqslant t_0 \geqslant 0 \;\forall\, x_0 \in \mathbb{R}^n$ $\|x(t, t_0, x_0)\| \leqslant M\|x_0\|$.

Denote $x_0 = (x_{0_1}, \ldots, x_{0_i}, \ldots, x_{0_n})^T$. Since $x(t, t_0, 0) = 0$, we have

$$\left\|\frac{\partial x(t, t_0, 0)}{\partial x_{0_i}}\right\| = \left\|\lim_{h\to 0} \frac{x(t, t_0, (0, \ldots, h, \ldots, 0)) - x(t, t_0, 0)}{h}\right\| \leqslant \lim_{h\to 0} \frac{M\|h\|}{\|h\|} = M.$$

So, for the transition matrix $\Phi(t, t_0, 0)$ of the variational equation $\dot{z} = f_x\big(t, x(t, t_0, 0)\big)z$, we have

$$\|\Phi(t, t_0, 0)\| = \left\|\frac{\partial x(t, t_0, 0)}{\partial x_0}\right\| \leqslant M.$$

It follows that the zero solution of (2.5) is globally uniformly stable. $\square$

Remark 2.1. Note that, the local version of Proposition 2.1 takes place as well. Namely, if the zero solution of (2.1) is locally uniformly (Lipshitz) stable in variation [17, Definition 1.3], [18, Definition 3.4], then the zero solution of (2.5) is locally uniformly stable. Indeed, if the zero solution of (2.1) is locally uniformly Lipshitz stable in variation, then, due to [17, Theorem 3.3], the zero solution of (2.1) is locally uniformly Lipshitz stable. Now, by [17, Theorem 3.4], the zero solution of (2.5) is locally uniformly Lipshitz stable. Since system (2.5) is linear, we get, by [17, Theorem 2.1], that the zero solution of (2.5) is locally uniformly stable.

Example 2.1. Let us give a simple example to show that one can lose the uniform character of attractiveness of solutions. Let us consider the following scalar differential equation:

$$\dot{x} = -\frac{x}{t+1}, \quad x \in \mathbb{R}, \quad t \geqslant 0.$$

Then,

$$x(t, t_0, x_0) = x_0 \frac{t_0 + 1}{t+1}.$$

We have

$$|x(t, t_0, x_0)| \leqslant |x_0|, \quad \forall\, t \geqslant t_0 \geqslant 0, \quad \forall\, x_0 \in \mathbb{R}^n.$$

The origin is globally uniformly stable in variation. Moreover, $\lim_{t\to\infty} |x(t, t_0, x_0)| = 0$. We have: for any $\epsilon > 0$, there exists $\delta = \epsilon$ such that if $|x_0| < \delta$, then $|x(t, t_0, x_0)| \leqslant |x_0| < \delta = \epsilon$ for $t \geqslant t_0 \geqslant 0$. This implies that the origin is asymptotically stable. But it is not uniformly attractive, and therefore, the origin is not uniformly asymptotically stable, because, for $T > 0$ and $t = t_0 + T$, one gets $x(t, t_0, x_0) = x_0 \frac{t_0+1}{t_0+T+1}$, which tends to x_0 as $t_0 \to \infty$.

Definition 2.2 (see [2, Definition 1.4]). The solution $x = 0$ of (2.1) is said to be *globally uniformly slowly growing in variation* if for every $\varepsilon > 0$ there exists a positive constant M, possibly depending on ε, such that

$$\|\Phi(t, t_0, x_0)\| \leqslant M e^{\varepsilon(t-t_0)} \quad \text{for all } t \geqslant t_0 \geqslant 0 \text{ and } x_0 \in \mathbb{R}^n.$$

Definition 2.3 (see [2, Definition 1.5]). The solution $x = 0$ of (2.1) is said to be *globally exponentially stable in variation* if there exist two positive constants λ_1 and λ_2, which are independent of the initial condition, such that

$$\|\Phi(t, t_0, x_0)\| \leqslant \lambda_1 e^{-\lambda_2(t-t_0)} \quad \text{for all } t \geqslant t_0 \geqslant 0 \text{ and } x_0 \in \mathbb{R}^n.$$

Example 2.2. As an example, consider the following scalar differential equation:

$$\dot{x} = -(t+1)x, \quad x(t_0) = x_0.$$

The solution with respect to the initial condition is given by:

$$x(t, t_0, x_0) = \Phi(t, t_0, x_0) x_0 \quad \text{where} \quad \Phi(t, t_0, x_0) = e^{-0.5\left((t+1)^2 - (t_0+1)^2\right)}.$$

It follows that, for all $t \geqslant t_0 \geqslant 0$ and $x_0 \in \mathbb{R}^n$,

$$\|\Phi(t, t_0, x_0)\| \leqslant e^{-0.5(t-t_0)(t+t_0+2)} \leqslant e^{-(t-t_0)}.$$

The above estimation implies that the zero solution is globally exponentially stable in variation.

Remark 2.2. Note that the exponential stability can be just local. As an example of system showing exponential stability but not global exponential stability, one can consider the following scalar differential equation: $\dot{x} = -x + x^2$. A simple computation shows that the solution with respect to (t_0, x_0) is given by:

$$x(t, t_0, x_0) = \frac{x_0 e^{-(t-t_0)}}{x_0 e^{-(t-t_0)} - x_0 + 1}.$$

The zero solution is exponentially stable but is not globally exponentially stable, because if we take $t = t_0$, $x_0 = 1$, then the solution $x(t, t_0, x_0)$ is equal to 1 when t goes to infinity.

Moreover, the exponential stability implies the asymptotic stability, but the converse is not true. As an example, we consider the differential equation: $\dot{x} = -x^3/2$. The solution of the initial value problem with the initial condition $x(t_0) = x_0$ is given by:

$$x(t, t_0, x_0) = x_0\big(x_0^2(t - t_0) + 1\big)^{-1/2}, \quad t \geqslant t_0 \geqslant 0.$$

It can be seen from the above solution that the zero solution is globally uniformly asymptotically stable but not exponentially stable.

Definition 2.4 (see [10]). The solution $x = 0$ of (2.1) is called *asymptotically stable in variation* if there exists a positive constant K such that, for every $t_0 \geqslant 0$ and all $t \geqslant t_0$,

$$\int_{t_0}^{t} \|\Phi(t, s, 0)\|\, ds \leqslant K.$$

The following result is established by Brauer [10, Theorem 1].

Proposition 2.2. *If the solution $x = 0$ of (2.1) is asymptotically stable in variation, then there exist constants $\alpha > 0$ and $\widetilde{M} > 0$ such that*

$$\int_{t_0}^{t} \sup_{\|x_0\| \leqslant \alpha} \|\Phi(t, s, x_0)\|\, ds \leqslant \widetilde{M},$$

for every sufficiently large t_0 and all $t \geqslant t_0 \geqslant 0$.

Note that, if the trivial solution $x = 0$ of (2.1) is asymptotically stable in variation, then [10, Lemma 3], for each $t_0 \geqslant 0$ and $\|x_0\| \leqslant \alpha$,

$$\lim_{t \to +\infty} \|\Phi(t, t_0, x_0)\| = 0.$$

Here, we have supposed that $f(t, 0) = 0,\ \forall t \geqslant 0$. For the studying stability of the perturbed system (2.2) in the case where $g(t) \neq 0$ for a certain $t \geqslant 0$, we shall study the asymptotic behavior of solutions in a neighborhood of the origin, in the sense that the solutions converge to a certain small ball B_r, $r > 0$, centered at the origin. Therefore, we introduce the notion of exponential stability of B_r (see [6, 7, 13–16, 20, 27]).

Definition 2.5. The set $B_r = \{x \in \mathbb{R}^n \mid \|x\| \leqslant r\}$ is said to be *globally uniformly exponentially stable with respect to the system* (2.2), if there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that solutions $x(t, t_0, x_0)$ of the system (2.2) satisfy the inequality

$$\|x(t, t_0, x_0)\| \leqslant \lambda_1 \|x_0\| e^{-\lambda_2 (t - t_0)} + r \qquad \forall\, t \geqslant t_0 \geqslant 0, \quad \forall x_0 \in \mathbb{R}^n \setminus B_r.$$

The factor λ_2 in the above definition will be named the convergence speed while factor λ_1 will be named the transient estimate.

Remark 2.3. It is worth to notice that, if we take $r = 0$ with $g(t) = 0,\ \forall t \geqslant 0$, then one deals with the standard concept of the global exponential stability of the origin viewed as an equilibrium point. It turns out that Definition 2.5 can be considered more general as the standard one, but by taking the radius small enough one can study the asymptotic behavior of the solutions near the origin which are not necessarily an equilibrium point. In this case, we shall study the asymptotic behavior of a small ball centered at the origin for $0 \leqslant \|x(t)\| - r$, $\forall t \geqslant t_0 \geqslant 0$, so that the initial conditions are taken outside the ball B_r. If r is small enough, then the trajectories approach a small neighborhood of the origin when t goes to infinity. In our case we consider systems which have disturbances (perturbations) where their presence in the model entails that the origin is not an equilibrium point and their boundedness gives that the radius r is strictly positive. Under these conditions the convergence of the solutions would be towards a small ball where the resulting radius is not optimal. It is possible that we can improve the asymptotic behavior by minimizing this radius. In certain situations we can give the solution directly. If not then one can use integral inequalities or the Lyapunov approach. The best we can do is to estimate r to be as small as possible. Moreover in our situation for the convergence of the solutions it's convenient to take initial conditions outside this ball. It turns out that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. Therefore, if the bound of perturbation term depends on a parameter ε with $\lim\limits_{\varepsilon\to 0} r(\varepsilon) = 0$, then the state will converges to the origin exponentially when t tends to infinity and $\varepsilon \to 0$.

Example 2.3. Taking into account the above Remark, one can illustrate the convergence of the solutions by different methods. As the first example, let us consider the following scalar differential equation: $\dot{x} = -x + e^{-2t}$. Here one can give the general solution explicitly. A simple computation shows that the solution with respect to (t_0, x_0) is given by:

$$x(t, t_0, x_0) = e^{-(t-t_0)}x_0 + e^{-(t+t_0)} - e^{-2t}.$$

It follows that

$$|x(t, t_0, x_0)| \leqslant |x_0|e^{-(t-t_0)} + 2, \quad t \geqslant t_0 \geqslant 0.$$

The last estimate shows a uniform convergence towards a certain neighborhood of the origin where the convergence speed $\lambda_2 = 1$ and the transient estimate $\lambda_1 = 1$. Therefore, if the initial conditions are taken outside the ball B_2, then the trajectories reach to this set when t goes to infinity.

Example 2.4. Note that the disturbance term can depend on the state; in the study of the convergence of solutions, the method of integral inequalities is a fundamental and very useful tool to have an idea for the asymptotic behavior of the system. The following example shows the convergence towards a neighborhood of the origin in the nonlinear case by using a suitable integral inequality. Let us consider the perturbed equation:

$$\dot{x}(t) = -\lambda x(t) + |x(t)|^p, \quad t \geqslant 0, \quad x \in \mathbb{R}, \quad \lambda > 0, \quad 0 < p < 1.$$

The solution with respect to the initial condition $\big(t_0, x(t_0)\big)$, $t_0 \geqslant 0$, is given by:

$$x(t) = e^{-\lambda(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{-\lambda(t-s)}|x(s)|^p\, ds.$$

It follows that

$$|x(t)| \leqslant e^{-\lambda(t-t_0)}|x(t_0)| + \int_{t_0}^{t} e^{-\lambda(t-s)}|x(s)|^p\, ds.$$

We have

$$e^{\lambda t}|x(t)| \leqslant e^{\lambda t_0}|x(t_0)| + \int_{t_0}^{t} e^{\lambda s}|x(s)|^p\, ds.$$

Let us set $u(t) = e^{\lambda t}|x(t)|$. Then we get

$$u(t) \leqslant u(t_0) + \int_{t_0}^{t} e^{(1-p)\lambda s}u^p(s)\, ds.$$

From [22, Theorem 21], we have

$$u(t) \leqslant \left\{ \big(u(t_0)\big)^{1-p} + (1-p) \int_{t_0}^{t} e^{(1-p)\lambda s} ds \right\}^{\frac{1}{1-p}}.$$

Thus,

$$u(t) \leqslant \left\{ \big(u(t_0)\big)^{1-p} + \frac{1}{\lambda}\big[e^{(1-p)\lambda t} - e^{(1-p)\lambda t_0}\big] \right\}^{\frac{1}{1-p}}.$$

Since, for any $q > 1$ and any $a, b \geqslant 0$,

$$(a+b)^q \leqslant 2^{q-1}(a^q + b^q),$$

then, by taking $q = \dfrac{1}{1-p} > 1$, we obtain:

$$u(t) \leqslant 2^{\frac{1}{1-p}-1} \left\{ \big(u(t_0)\big) + \left(\frac{1}{\lambda}\big[e^{(1-p)\lambda t} - e^{(1-p)\lambda t_0}\big]\right)^{\frac{1}{1-p}} \right\}.$$

Then

$$e^{\lambda t}|x(t)| \leqslant 2^{\frac{p}{1-p}} \Big(e^{\lambda t_0}|x(t_0)|\Big) + \left(\frac{2^p}{\lambda}\right)^{\frac{1}{1-p}} \big[e^{(1-p)\lambda t} - e^{(1-p)\lambda t_0}\big]^{\frac{1}{1-p}}. \tag{2.6}$$

We have

$$\big[e^{(1-p)\lambda t} - e^{(1-p)\lambda t_0}\big]^{\frac{1}{1-p}} \leqslant \big(e^{(1-p)\lambda t}\big)^{\frac{1}{1-p}} = e^{\lambda t}. \tag{2.7}$$

From (2.6) and (2.7), it follows that

$$|x(t)| \leqslant 2^{\frac{p}{1-p}} |x(t_0)| e^{-\lambda(t-t_0)} + \left(\frac{2^p}{\lambda}\right)^{\frac{1}{1-p}}.$$

So, we obtain an estimate on the solution as in Definition 2.5.

Furthermore, we can study the convergence of solutions to a small neighborhood of the origin by using the Lyapunov approach, in this sense. Let us consider the following example.

Example 2.5. For the following scalar equation, we will use Lyapunov techniques as an effective tool especially for the study of convergence of solutions, since we can take the Lyapunov function of the system without disturbances as a Lyapunov function candidate for the entire system with the disturbance. Let us consider the following perturbed differential equation

$$\dot{x} = -x + \varepsilon \frac{x}{x^2+1} e^{-t}, \quad x \in \mathbb{R}, \quad \varepsilon > 0.$$

Take $V(t,x) = x^2$. By taking the derivative along the trajectories, we obtain:

$$\dot{V}(t,x) = -2x^2 + 2\varepsilon \frac{x^2}{x^2+1} e^{-t} \leqslant -2x^2 + 2\varepsilon e^{-t}.$$

Hence,

$$\dot{V}(t,x) \leqslant -2V(t,x) + 2\varepsilon. \tag{2.8}$$

Let us find an estimation for $V\big(t, x(t)\big)$ from the differential inequality (2.8). Denote $z(t) = V\big(t, x(t)\big)$. Let the initial state $z(t_0)$, $t_0 \geqslant 0$, satisfy condition

$$z(t_0) \geqslant \varepsilon > 0. \tag{2.9}$$

From (2.8), it follows that
$$\dot z(t) \leqslant -2z(t) + 2\varepsilon, \quad t \geqslant t_0. \tag{2.10}$$
Replace $u(t) := z(t) - \varepsilon$. Hence, from (2.10), it follows that
$$\dot u(t) \leqslant -2u(t), \quad t \geqslant t_0, \tag{2.11}$$
and, from (2.9), it follows that $u(t_0) \geqslant 0$. Denote $u(t) := v(t)e^{-2(t-t_0)}$. Then, from (2.11), it follows that $\dot v(t) \leqslant 0$, $t \geqslant t_0$. Hence, $v(t) \leqslant v(t_0) = u(t_0)$. Therefore,
$$u(t) = v(t)e^{-2(t-t_0)} \leqslant v(t_0)e^{-2(t-t_0)} = u(t_0)e^{-2(t-t_0)}.$$
So,
$$z(t) \leqslant \big(z(t_0) - \varepsilon\big)e^{-2(t-t_0)} + \varepsilon, \quad t \geqslant t_0. \tag{2.12}$$
Now, using the fact that $V(t,x) = x^2$, we obtain from (2.12) that
$$x^2(t) \leqslant \big(x^2(t_0) - \varepsilon\big)e^{-2(t-t_0)} + \varepsilon, \quad t \geqslant t_0. \tag{2.13}$$
Let the initial condition $x(t_0)$ is taken outside $[-\sqrt{\varepsilon}, +\sqrt{\varepsilon}]$. Hence, $x^2(t_0) - \varepsilon > 0$, and, in particuar, (2.9) holds. Then, from (2.13), it follows that
$$|x(t)| \leqslant (x^2(t_0) - \varepsilon)^{1/2}e^{-(t-t_0)} + \sqrt{\varepsilon} \leqslant \|x(t_0)\|e^{-(t-t_0)} + \sqrt{\varepsilon}, \quad t \geqslant t_0.$$
So, we get the estimate on the solution as in Definition 2.5, and this estimation gives the exponential convergence of the solution toward the ball $B_{\sqrt{\varepsilon}} = [-\sqrt{\varepsilon}, +\sqrt{\varepsilon}]$. Here, the bound of perturbation term depends on the parameter ε with $\lim\limits_{\varepsilon\to 0} r(\varepsilon) = 0$, then the state will converges to the origin exponentially when t tends to infinity and $\varepsilon \to 0$.

In the sequel, we will recall the definitions of comparison functions. For time-varying systems, the authors in [32] studied how definitions of uniform global asymptotic stability that have been employed over the years in a variety of monographs and publications lack "uniformity". Uniform global attraction and uniform local stability are sometimes combined to make uniform global asymptotic stability. In order to study the asymptotic behavior of system (2.2), we will use the $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions which perfectly and rigorously characterize the uniform asymptotic stability (see [25, 29, 36]).

- A continuous function $\alpha : [0, +\infty) \to [0, +\infty)$ is said to *belong to class* $\mathcal{K}$ [29, Definition 4.2] if it is strictly increasing and $\alpha(0) = 0$. It is said to *belong to class* $\mathcal{K}_\infty$, if $\alpha(r) \to +\infty$ as $r \to +\infty$.

- A continuous function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to *belong to class* $\mathcal{KL}$ [29, Definition 4.3] if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to +\infty$.

By [29, Lemma 4.5], the zero equilibrium of system (2.2) with $g(t) \equiv 0$ is globally uniformly asymptotically stable if and only if there exists a class $\mathcal{KL}$ function β such that $\|x(t, t_0, x_0)\| \leqslant$ $\leqslant \beta(\|x_0\|, t - t_0), \quad \forall\, t \geqslant t_0 \geqslant 0,\ \forall x_0 \in \mathbb{R}^n$.

The asymptotic behavior of the solutions of (2.2) can be studied in a neighborhood of the origin, in this case the solutions converge to a certain small ball.

Definition 2.6. The ball B_r is said to be *globally uniformly asymptotically stable with respect to the system* (2.2), if there exists a class $\mathcal{KL}$ function β such that the solutions $x(t, t_0, x_0)$ of the system (2.2) satisfy the inequality

$$\|x(t, t_0, x_0)\| \leqslant \beta(\|x_0\|, t - t_0) + r, \qquad \forall\, t \geqslant t_0 \geqslant 0, \quad \forall x_0 \in \mathbb{R}^n \setminus B_r.$$

When the origin is not an equilibrium point, one can examine the asymptotic behavior of the solution with regard to a small ball centered at the origin. There are many results which relate the asymptotic stability of the zero solution of the unperturbed system to that the zero solution of the perturbed equation. The relation can be studied through a slight variant of the nonlinear variation of constants formula. Let $\widetilde{x}(t, t_0, x_0)$ be the solution of (2.2) passing through (t_0, x_0) and $x(t, t_0, x_0)$ be the solution of (2.1) passing through (t_0, x_0). Then (see, e.g., [1,9]), we have

$$\widetilde{x}(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^{t} \Phi\big(t, s, \widetilde{x}(s, t_0, x_0)\big) g(s)\, ds.$$

So, by imposing some restrictions on the term of perturbation, one can reach conclusions on the stability of the perturbed system.

In the sequel, we give an example of class of scalar perturbed linear differential equations where solutions satisfy an estimation as the one given in Definition 2.6. In fact, the qualitative analysis of solutions of linear differential equations and their perturbed linear differential equations is crucial for addressing a wide range of practical issues in the fields of mechanical, electrical, control, and economic engineering. As a result, several authors have investigated many questions along these lines and highlighted a variety of attributes where they suppose in general that the origin is an equilibrium point.

Example 2.6. Let us consider the following scalar linear system:

$$\dot{x} = a(t)x + g(t), \quad t \geqslant 0,$$

where $a(\cdot)$ is continuous and $g(\cdot)$ is a continuous bounded function. Then, one has

$$x(t, t_0, x_0) = x_0 \exp\Big(\int_{t_0}^{t} a(s)\, ds\Big) + \int_{t_0}^{t} g(\tau) \exp\Big(\int_{\tau}^{t} a(s)\, ds\Big)\, d\tau.$$

Note that, the concept of stability of the nominal unperturbed scalar linear time-varying equation is related to the transition matrix $\Phi(t, t_0) = \exp\Big(\int_{t_0}^{t} a(\tau)\, d\tau\Big)$. The asymptotic stability is characterized by the fact that

$$\lim_{t \to +\infty} \int_{t_0}^{t} a(\tau)\, d\tau = -\infty.$$

The exponential stability is characterized by the fact that

$$\limsup_{t \to +\infty} \frac{1}{t - t_0} \int_{t_0}^{t} a(\tau)\, d\tau < 0.$$

So, if the nominal unperturbed system is exponentially stable at the origin and the estimation

$$\int_{t_0}^{t} g(\tau) \exp\Big(\int_{\tau}^{t} a(s)\, ds\Big) d\tau \leqslant r$$

holds for some $r > 0$, then one can obtain an estimation on the solutions in presence of the term of perturbation, where the radius r of the ball B_r depends on the size of the bound of the function g. Indeed, to obtain an estimation as in Definition 2.6, it suffices to suppose that the nominal system is globally uniformly asymptically stable, which is characterized by the fact that $\int_{t_0}^{t} a(\tau)\,d\tau \leqslant \lambda_1 - \lambda_2(t-t_0)$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\forall t \geqslant t_0 \geqslant 0$. In this case, with $\beta(|x_0|, t-t_0) = e^{\lambda_1}|x_0| \exp\big(-\lambda_2(t-t_0)\big)$ one has $|x(t,t_0,x_0)| \leqslant \beta(|x_0|, t-t_0) + r$.

Example 2.7. Note that, we can take in the example above a nominal unperturbed system which is non-linear, instead of a linear one. As an example, we consider the differential equation:

$$\dot{x} = -e^t x^3. \tag{2.14}$$

The solution is given by

$$x(t,t_0,x_0) = \frac{x_0}{\sqrt{1+2x_0^2(e^t - e^{t_0})}}, \qquad t \geqslant t_0 \geqslant 0. \tag{2.15}$$

We have: for any $x_0 \in \mathbb{R}$ and $t \geqslant t_0 \geqslant 0$,

$$1 + 2x_0^2(e^t - e^{t_0}) = 1 + 2x_0^2 e^{t_0}(e^{t-t_0} - 1) \geqslant 1 + 2x_0^2(e^{t-t_0} - 1). \tag{2.16}$$

Set $\beta(r,s) = \dfrac{r}{\sqrt{1+2r^2(e^s-1)}}$, $r \geqslant 0$, $s \geqslant 0$. Then $\beta(0,s) = 0$,

$$\frac{\partial \beta}{\partial r} = \frac{1}{\big(1+2r^2(e^s-1)\big)^{3/2}} > 0.$$

Hence, $\beta(r,s)$ is strictly increasing in r. Next,

$$\frac{\partial \beta}{\partial s} = -\frac{r^3 e^s}{\big(1+2r^2(e^s-1)\big)^{3/2}} < 0.$$

Hence, $\beta(r,s)$ is strictly decreasing in s. Moreover, $\beta(r,s) \to 0$ as $s \to +\infty$. Therefore, it belongs to class $\mathcal{KL}$.

From (2.15) and (2.16), it follows that

$$|x(t,t_0,x_0)| \leqslant \frac{|x_0|}{\sqrt{1+2x_0^2(e^t - e^{t_0})}} \leqslant \frac{|x_0|}{\sqrt{1+2x_0^2(e^{t-t_0} - 1)}} = \beta(|x_0|, t-t_0).$$

It follows that the zero solution of system (2.14) is globally uniformly asymptotically stable.

Calculating $\Phi(t,t_0,x_0)$, we get:

$$\Phi(t,t_0,x_0) = \big(1+2x_0^2(e^t - e^{t_0})\big)^{-3/2}.$$

Thus, $|\Phi(t,t_0,x_0)| \leqslant 1, \forall\, t \geqslant t_0 \geqslant 0$ Then, by considering a disturbance similar to the one given in the previous example, we can arrive at a similar estimate on the trajectories.

The perturbations were represented by an additive term on the right-hand side of the equation of state (2.2) and the origin was not supposed to be an equilibrium point of the system. Based on the stability of the nominal system (2.1), which had the origin as its equilibrium point, we cannot expect that the solution of the perturbed system will approach the origin as t goes to infinity. The best we can hope for is that for a small perturbation term the solution tends to a small set containing the origin. However, the desired state of the system may be mathematically unstable, but the system may oscillate close enough to that state that the performance is considered acceptable. In a typical situation we do not know g but just some information about it, for example the upper bound on the norm of g. A natural approach is to solve this problem by using a Lyapunov function for the nominal system as a Lyapunov function candidate for the whole system.

Stability analysis for linear time-varying systems is of increasing interest in theory. One reason is the growing importance of adaptive controllers for which the underlying closed-loop adaptive system often is time-varying and linear which can be modeled as

$$\dot{x} = A(t)x, \tag{2.17}$$

where A is an $n \times n$-matrix whose entries are real-valued piecewise continuous functions of $t \in \mathbb{R}^+$. The space of solutions has dimension n. A basis of the space of solutions of this system, i. e., the set $\{x_1, \ldots, x_n\}$ of linearly independent solutions, is called *a fundamental set of solutions*. The matrix $\Psi(t) = [x_1(t) \ \ldots \ x_n(t)]$, whose columns are the basis vectors of the solution space, is called *a fundamental matrix*. A fundamental matrix is a solution to the matrix equation

$$\dot{\Psi}(t) = A(t)\Psi(t)$$

and conversely, any nonsingular solution of the above system is a fundamental matrix of the linear system. Let $\Psi(t)$ be a fundamental matrix. Then

$$\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0), \quad t \geqslant t_0,$$

is called *the state transition matrix*. Notice that the above definition is consistent in the sense that $\Phi(t, t_0)$ is uniquely defined by $A(t)$ and independent of the particular choice of $\Psi(t)$ (see [19, Ch. II, Sect. 2]).

We have a characterization for uniform asymptotic stability. The trivial solution of the linear system (2.17) is globally uniformly asymptotically stable if and only if it is exponentially stable [29, Sect. 4.6] that is there exist positive constants k and γ such that

$$\|\Phi(t, t_0)\| \leqslant ke^{-\gamma(t-t_0)}, \quad \forall t \geqslant t_0 \geqslant 0.$$

The solution of (2.17) with the initial condition $x(t_0) = x_0$ is $x(t) = \Phi(t, t_0)x_0$, $t \geqslant t_0$. This formula can be directly checked using the definition relation $\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0)$, $t \geqslant t_0$. It shows that the state transition matrix is a linear transformation that maps the initial condition x_0 into the state x at time t. If the system is time-invariant ($A(t) = A$), then $\Phi(t, t_0) = \exp\big(A(t - t_0)\big)$. If $A(t)$ is not constant and the matrices $A(t)$ and $A(s)$ commute for all (t, s), then $\Phi(t, t_0) = \exp\Big(\int_{t_0}^{t} A(s)\,ds\Big)$.

Let us consider the time-varying system (2.1) in the case where the nominal system is linear, namely, $f(t, x) = A(t)x$, and the perturbation term in system (2.2) is taken as $g(t, x)$ instead of $g(t)$:

$$\dot{x} = A(t)x + g(t, x). \tag{2.18}$$

Suppose that $A(t)$ is an $(n \times n)$ continuous and bounded matrix, $g(t, x)$ is a continuous function, and there exists a nonnegative continuous function $\zeta(t)$ such that

$$\|g(t, x)\| \leqslant \zeta(t), \quad x \in \mathbb{R}^n, \quad t \geqslant 0.$$

We suppose that the bounds of the nonlinearities satisfy the following condition:

$$\text{either} \quad \int_0^{+\infty} \zeta(s)\,ds < +\infty \quad \text{or} \quad \lim_{t\to\infty} \zeta(t) = 0.$$

If the function $\zeta(t)$ satisfies one of the last two conditions, then, for any given $\lambda > 0$ and for any fixed $t_0 \geqslant 0$,

$$\lim_{t\to\infty} e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s}\zeta(s)\,ds = 0.$$

This fact was shown in [27] for the case $t_0 = 0$. It is clear that this is true for any fixed $t_0 \geqslant 0$.

Further, if the nominal system is uniformly asymptotically stable, then the associated transition matrix satisfies condition

$$\|\Phi(t,t_0)\| \leqslant \lambda_1 e^{-\lambda_2(t-t_0)}, \quad \forall\, t \geqslant t_0 \geqslant 0,$$

for some $\lambda_1 > 0$, $\lambda_2 > 0$. The solution of system (2.18) with an initial condition $x(t_0) = x_0$ $(t_0 \geqslant 0)$ can be written as

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)g\big(s,x(s)\big)\,ds, \quad t \geqslant t_0.$$

Then,

$$\|x(t)\| \leqslant \|\Phi(t,t_0)\|\;\|x_0\| + \int_{t_0}^{t} \|\Phi(t,s)\|\;\big\|g\big(s,x(s)\big)\big\|\,ds, \quad t \geqslant t_0.$$

It follows that

$$\|x(t)\| \leqslant \lambda_1\|x_0\|e^{-\lambda_2(t-t_0)} + \int_{t_0}^{t} \lambda_1 e^{-\lambda_2(t-s)}\zeta(s)\,ds, \quad t \geqslant t_0.$$

Thus,

$$\|x(t)\| \leqslant \lambda_1\|x_0\|e^{-\lambda_2(t-t_0)} + \lambda_1 e^{-\lambda_2 t}\int_{t_0}^{t} e^{\lambda_2 s}\zeta(s)\,ds, \quad t \geqslant t_0.$$

Since

$$\lim_{t\to\infty} e^{-\lambda_2 t}\int_{t_0}^{t} e^{\lambda_2 s}\zeta(s)\,ds = 0,$$

then there exists $\widetilde{\zeta} > 0$ such that

$$e^{-\lambda_2 t}\int_{t_0}^{t} e^{\lambda_2 s}\zeta(s)\,ds \leqslant \widetilde{\zeta}, \quad \forall t \geqslant t_0.$$

It follows that

$$\|x(t)\| \leqslant \lambda_1\|x_0\|e^{-\lambda_2(t-t_0)} + \lambda_1\widetilde{\zeta}, \quad t \geqslant t_0.$$

Hence, as in Definition 2.5, the ball B_r with $r = \lambda_1\widetilde{\zeta}$ is globally uniformly exponentially stable with respect to the system (2.18).

Note that (see [27]): if the function $\zeta(t)$ satisfies $\displaystyle\int_0^{+\infty} \zeta(s)\,ds \leqslant \xi < +\infty$, then

$$e^{-\lambda_2 t}\int_{t_0}^{t} e^{\lambda_2 s}\zeta(s)\,ds \leqslant e^{-\lambda_2 t}\int_{0}^{t} e^{\lambda_2 s}\zeta(s)\,ds \leqslant \frac{\xi}{\lambda_2}, \qquad \forall t \geqslant t_0 \geqslant 0;$$

if the function $\zeta(t)$ satisfies $\|\zeta(t)\| \leqslant \eta$, $\forall t \geqslant 0$, then $e^{-\lambda_2 t}\displaystyle\int_{t_0}^{t} e^{\lambda_2 s}\zeta(s)\,ds \leqslant \frac{\eta}{\lambda_2}$, $\forall t \geqslant t_0 \geqslant 0$.

§ 3. Lyapunov approach

Lyapunov's direct method allows one to determine the stability of a system without explicitly integrating the differential equation. This method is a generalization of the idea that if there is an appropriate energy function in a system, then we can study the rate of change of the energy of the system to a certain stability. To make this precise, we need the following definitions.

Consider a continuous function $V\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$. V is said to be *globally Lipschitzian in* x (uniformly in $t \in \mathbb{R}^+$) if

$$|V(t,x) - V(t,y)| \leqslant K\|x-y\|$$

for some $K>0$ and for all $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$. Corresponding to V we define the Dini derivative D^+V with respect to system (2.1) by

$$D^+V_f(t,x) = \limsup_{h\to 0^+} \frac{1}{h}\big(V(t+h, x+hf(t,x)) - V(t,x)\big),$$

called the upper Dini derivative of $V(.,.)$ along the trajectory of (2.1). Let $x(t)$ be a solution of (2.1). Denote by $V'\big(t,x(t)\big)$ the upper right-hand derivative of $V\big(t,x(t)\big)$, i. e.,

$$V'\big(t,x(t)\big) = \limsup_{h\to 0^+} \frac{1}{h}\Big(V\big(t+h, x(t+h)\big) - V\big(t,x(t)\big)\Big).$$

If $V(t,x)$ is continuous in t and Lipschitzian in x (uniformly in t) with the Lipschitz constant $K>0$, then (see [37, p. 3]),

$$D^+V_f\big(t,x(t)\big) = V'\big(t,x(t)\big).$$

Note that, in case when the function V is differentiable, the derivative with respect to time along the trajectories of system (2.1) is given by:

$$\frac{d}{dt}V(t,x) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x).f(t,x),$$

in this case, we have

$$\frac{d}{dt}V(t,x) = V'(t,x) = D_f^+V(t,x).$$

Suppose that the nominal system (2.1) has a uniformly asymptotically/exponentially stable equilibrium point at the origin, then under some sufficient conditions on the perturbation term we can study the asymptotic behavior of the solutions of (2.2). A natural approach is to use the Lyapunov function $V(t,x)$ for the unperturbed system as a Lyapunov function candidate for (2.2). Note that, one can reach the conclusion about the definiteness of $\dot{V}(t,x)$ by imposing some restrictions on $g(t)$, using the Lyapunov function of the form

$$\mathcal{V}(t,x) = V(t,x) + \Psi(t,x),$$

where the function $\Psi(t,x)$ is defined by the expression:

$$\Psi(t,x) = \int_t^{+\infty} \frac{\partial V}{\partial x}\big(s,\phi(s,t,x)\big)\cdot g(s)\,ds.$$

Here $\phi(s,t,x)$ is the solution of the unperturbed system (2.1) such that $\phi(t,t,x) = x$, and the auxiliary function $\Psi(t,x)$ is chosen so that the function $\mathcal{V}(t,x)$ is positive definite, and its derivative along the trajectories of (2.2) is negative definite. Naturally, the choice of $\Psi(t,x)$ depends on the perturbation term $g(t)$ and its smoothness is given under some restrictions on the dynamics of the system (see [5]). The strict Lyapunov functions are of great importance in the study of the stability of systems, and are a key tool for robustness analysis. In general, it is more difficult to construct strict Lyapunov functions for time-varying systems than it is for time-invariant systems. Using the above idea for constructing strict Lyapunov functions for time-varying systems, the class of perturbed systems where the nominal system is linear is considered by [4].

Suppose that the nominal system (2.1) has a uniformly asymptotically/exponentially stable in variation equilibrium point at the origin with $V(t,x)$ as a Lyapunov function candidate. Such Lyapunov function should satisfy the following assumptions [2]:

$(\mathcal{H}_1)$ $V(t,x)$ is defined and continuous on $\mathbb{R}^+ \times \mathbb{R}^n$;

$(\mathcal{H}_2)$ $\|x\| \leqslant V(t,x) \leqslant K_1\|x\|$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, for some $K_1 \geqslant 1$;

$(\mathcal{H}_3)$ $|V(t,x) - V(t,y)| \leqslant K_2\|x-y\|$ for all $(t,x),(t,y) \in \mathbb{R}^+ \times \mathbb{R}^n$, for some $K_2 > 0$.

In [10, 11] and [18], some properties and converse theorems for the kinds of stability in variation sense are given. In the sequel, we recall some of them.

Let the trivial solution of (2.1) be globally uniformly stable in variation. Then (see [2]) there exists a function $V(t,x)$ which satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and the following property:

$$D_f^+V(t,x) \leqslant 0, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Note that, for the existence of such function, it suffices to take as a Lyapunov function candidate (see [2]):

$$V(t,x) = \sup_{s \geqslant 0} \|\phi(t+s,t,x)\|, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Furthermore, let the trivial solution of (2.1) be globally uniformly slowly growing in variation. Then (see [2]) there exists a function $V(t,x)$ which satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and the following estimation, for some $\varepsilon > 0$:

$$D_f^+V(t,x) \leqslant \varepsilon V(t,x), \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Here, a Lyapunov function candidate can be taken as:

$$V(t,x) = \sup_{s \geqslant 0} \|\phi(t+s,t,x)\| e^{-\varepsilon s}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

In term of exponential stability, if the trivial solution of (2.1) is globally exponentially stable in variation, then there exists a function $V(t,x)$ which satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and the following inequality, for some $\widetilde{\alpha} > 0$ (see [2]):

$$D_f^+V(t,x) \leqslant -\widetilde{\alpha} V(t,x), \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

In this case, it suffices to take (see [2]):

$$V(t,x) = \sup_{s \geqslant 0} \|\phi(t+s,t,x)\| e^{\widetilde{\alpha} s}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

In any study of stability of dynamical systems under perturbations, the asymptotic equivalence of two systems is one of the most important concepts. It can be used to study the robustness of the unperturbed system or to explore whether the behavior of a complex system can be determined by the behavior of a simpler system. The basic idea in studies of stability of nonlinear systems, apparently, is to decompose the system into isolated subsystems and the systems connecting them, and then determine the stability of the original system from the asymptotic behavior of the subsystems. However, in general, the original system and the subsystems may not be asymptotically equivalent, which may produce misleading results.

Some sufficient conditions can be obtained for the uniform Lipschitz stability of the system (2.1) $\dot{x} = f(t,x)$.

Suppose that there exists differentiable $V(t,x)$ satisfying the following assumptions.

$(\mathcal{A}_1)$ There exist two functions α_1 and α_2 of class $\mathcal{K}$ such that

$$\alpha_1\big(\|x\|\big) \leqslant V(t,x) \leqslant \alpha_2\big(\|x\|\big), \quad \forall x \in \mathbb{R}^n, \quad \forall t \geqslant t_0 \geqslant 0,$$

and $\dfrac{{\alpha_1}^{-1}\big(\alpha_2(s)\big)}{s}$ is bounded for $s \geqslant 0$ that is there exists a constant $\zeta \geqslant 0$ such that

$$\frac{{\alpha_1}^{-1}\big(\alpha_2(s)\big)}{s} \leqslant \zeta, \quad \forall s \geqslant 0.$$

$(\mathcal{A}_2)$ $V'(t,x) = D_f^+ V(t,x) \leqslant 0, \quad \forall x \in \mathbb{R}^n, \quad t \geqslant t_0 \geqslant 0.$

Proposition 3.1. *Suppose that there exists a Lyapunov function $V(t,x)$ that satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Then the solution $x(t,t_0,x_0)$ of (2.1) satisfies the following inequality for all $t \geqslant t_0 \geqslant 0$:*

$$\|x(t,t_0,x_0)\| \leqslant \zeta \|x_0\|.$$

Indeed, from $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, one gets

$$\alpha_1\big(\|x(t,t_0,x_0)\|\big) \leqslant V\big(t, x(t,t_0,x_0)\big) \leqslant V(t_0,x_0) \leqslant \alpha_2\big(\|x_0\|\big), \quad \forall x_0 \in \mathbb{R}^n, \quad \forall t \geqslant t_0 \geqslant 0.$$

Thus,

$$\|x(t,t_0,x_0)\| \leqslant {\alpha_1}^{-1}\big(\alpha_2(\|x_0\|)\big) \leqslant \zeta \|x_0\|.$$

This implies that the solution $x = 0$ of (2.1) is uniformly Lipschitz stable [17, Definition 1.1].

§ 4. Asymptotic behavior of solutions

In this section, we give some new results on asymptotic behavior and growth properties of the solutions of (2.2) under some restrictive conditions on the perturbation term based on the following well known comparison lemma [29, Lemma 3.4].

Lemma 4.1. *Consider a scalar differential equation:*

$$\dot{u}(t) = h(t,u), \quad u(t_0) = u_0, \tag{4.1}$$

where $h(t,u)$ is continuous in t and locally Lipschitz in u, for all $t \geqslant 0$ and all $u \in J \in \mathbb{R}$. Let $[t_0, \vartheta)$ be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, \vartheta)$. Let $v(t)$ be a continuous function such that

$$D^+ v(t) \leqslant h\big(t, v(t)\big), \quad v(t_0) \leqslant u_0,$$

with $v(t) \in J$ for all $t \in [t_0, \vartheta)$. Then, $v(t) \leqslant u(t)$ for all $t \in [t_0, \vartheta)$.

This lemma can provide an estimation on $V\big(t, x(t)\big)$ from some bounds on $D_f^+ V(t,x)$. Let $x(t) = x(t,t_0,x_0)$ be a solution of (2.1) existing for $t \geqslant t_0 \geqslant 0$. Suppose that $V(t,x)$ is continuous in t and globally Lipschitzian in x (uniformly in $t \in \mathbb{R}_+$) and satisfies the inequality:

$$D_f^+ V(t,x) \leqslant h\big(t, V(t,x)\big)$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then, if $V(t_0,x_0) \leqslant u_0$, then we have

$$V\big(t, x(t)\big) \leqslant u(t), \quad \text{for } t \geqslant t_0 \geqslant 0. \tag{4.2}$$

The origin may not be an equilibrium point of the perturbed system (2.2). We can no longer study the stability of the origin as an equilibrium point, and we should not expect the solution of the perturbed system to approach the origin as t tends to infinity. The best we can hope that for a small perturbation term the solution will approach to a small set containing the origin in the case where $g(t) \neq 0$ for some $t > 0$. We first present the following result, which gives an estimate of the solutions of the perturbed system when we suppose that the nominal system is globally uniformly stable in variation.

Now, for asymptotic convergence, we need the following lemma (see [26, Lemma 2.4]).

Lemma 4.2. *Let $y\colon [0,+\infty) \longrightarrow [0,+\infty)$ be a differentiable function, α be a class $\mathcal{K}_\infty$ function and c be a positive real number. Assume that for all $t \in [0,+\infty)$ we have*

$$\dot{y}(t) \leqslant -\alpha\big(y(t)\big) + c.$$

Then, there exists a class $\mathcal{KL}$ function β_α such that

$$y(t) \leqslant \alpha^{-1}(2c) + \beta_\alpha\big(y(0), t\big).$$

Theorem 4.1. *Suppose that there exists a Lyapunov function $V(t,x)$ satisfying conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, and there exist a function $\alpha \in \mathcal{K}_\infty$ and a number $c > 0$ such that the following inequality holds:*

$$D_f^+ V(t,x) \leqslant -\alpha\big(\|x\|\big) + c, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{4.3}$$

Then, the solutions of (2.1) converge globally uniformly asymptotically to a certain ball centered at the origin.

P r o o f. From (4.3), by using $(\mathcal{H}_2)$, we get

$$D_f^+ V(t,x) \leqslant -\alpha\left(\frac{1}{K_1} V(t,x)\right) + c.$$

Thus,

$$D_f^+ V(t,x) \leqslant -\alpha_1\big(V(t,x)\big) + c, \tag{4.4}$$

where $\alpha_1(r) = \alpha\big(r/K_1\big) \in \mathcal{K}_\infty$. Let $x(t) = x(t,t_0,x_0)$ be a solution of (2.1) and $V(t_0,x_0) = u_0$, $u_0 \geqslant 0$. Consider the equation (4.1) where

$$h(t,u) \equiv h(u) = -\alpha_1(u) + c. \tag{4.5}$$

Consider the solution $u(t,t_0,u_0)$ of (4.1), (4.5). Since equation (4.1), (4.5) is time-invariant, we have

$$u(t,t_0,u_0) = u(t-t_0,0,u_0) = u(\tau,0,u_0) \tag{4.6}$$

where $\tau = t - t_0$.

Let us apply Lemma 4.2 to the function $y(\tau) = u(\tau,0,u_0)$. Thus,

$$u(\tau,0,u_0) \leqslant \beta_{\alpha_1}\big(u_0,\tau\big) + \alpha_1^{-1}(2c)$$

where β_{α_1} is a function of the class $\mathcal{KL}$. By taking into account (4.2) and (4.6), it follows that

$$V\big(t, x(t,t_0,x_0)\big) \leqslant u(t,t_0,u_0) = u(\tau,0,u_0) \leqslant \beta_{\alpha_1}\big(V(t_0,x_0), t-t_0\big) + \alpha_1^{-1}(2c).$$

Hence, from $(\mathcal{H}_2)$,

$$\|x(t,t_0,x_0)\| \leqslant \beta_{\alpha_1}\big(K_1\|x_0\|, t-t_0\big) + \alpha_1^{-1}(2c).$$

Set $\widetilde{\beta}(r,s) := \beta_{\alpha_1}(K_1 r, s)$. Then, $\widetilde{\beta} \in \mathcal{KL}$ and

$$\|x(t,t_0,x_0)\| \leqslant \widetilde{\beta}\big(\|x_0\|, t-t_0\big) + \alpha_1^{-1}(2c).$$

Thus, the ball B_δ with $\delta = \alpha_1^{-1}(2c)$ is globally uniformly asymptotically stable. $\square$

Corollary 4.1. *Suppose that there exists a Lyapunov function $V(t,x)$ satisfying conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, and there exist functions $\alpha \in \mathcal{K}_\infty$ and $c:(0,+\infty)\to(0,+\infty)$ such that the following inequality holds:*

$$D_f^+V(t,x) \leqslant -\alpha\big(\|x\|\big) + c(\varepsilon), \quad \forall (t,x)\in\mathbb{R}^+\times\mathbb{R}^n, \quad \varepsilon>0.$$

Then, the solutions of (2.1) *converge globally uniformly asymptotically to the ball $B_{\delta(\varepsilon)}$ with*

$$\delta(\varepsilon) = \alpha_1^{-1}\big(2c(\varepsilon)\big).$$

Note that, if $\lim\limits_{\varepsilon\to+0} c(\varepsilon)=0$, then the solutions approach the origin.

The stability analysis of perturbed differential equations is generally based on the stability of the nominal system, provided that the size of the perturbation is known, as is knowledge of the upper bound of the perturbation term that may arise from modeling errors or perturbations. By utilizing the given form of the equations, one can study the asymptotic behaviors of the system without explicit knowledge of the solutions.

Let us consider the following perturbed system:

$$\dot{x} = f(t,x) + g(t,x) \tag{4.7}$$

where $t\in\mathbb{R}^+$; $f\colon \mathbb{R}^+\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function, locally Lipschitz with respect to x such that $f(t,0)=0$, $\forall t\geqslant 0$; $g\colon \mathbb{R}^+\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function, which represents the disturbance term, such that

$$\|g(t,x)\| \leqslant \lambda(t,x)\alpha_0(\|x\|) + \xi(t,x), \quad \forall\, t\in\mathbb{R}_+, \quad \forall\, x\in\mathbb{R}^n, \tag{4.8}$$

where $\lambda(\cdot),\ \xi(\cdot)\in C(\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^+)$ and $\alpha_0\in\mathcal{K}_\infty$.

Now, with the above appropriate condition made on the perturbation term $g(t,x)$, we examine the behavior of solutions to the perturbed system (4.7) with the properties given in $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$.

Theorem 4.2. *Suppose that there exists a Lyapunov function $V(t,x)$ such that conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are fulfilled, and there exists $\alpha\in\mathcal{K}_\infty$ such that the following inequality holds:*

$$D_f^+V(t,x) \leqslant -\alpha(\|x\|), \quad \forall (t,x)\in\mathbb{R}^+\times\mathbb{R}^n. \tag{4.9}$$

Suppose that the perturbation term satisfies inequality (4.8)*, where*

$$\sup_{t\in\mathbb{R}^+\times\mathbb{R}^n} \lambda(t,x) \leqslant \lambda_0 < +\infty, \tag{4.10}$$

$$\sup_{t\in\mathbb{R}^+\times\mathbb{R}^n} \xi(t,x) \leqslant \kappa < +\infty, \tag{4.11}$$

and α_0 is selected such that, for some $l\in(0,1)$, for all $r\geqslant 0$,

$$\alpha_0(r) \leqslant \frac{l}{K_2\lambda_0}\alpha(r). \tag{4.12}$$

Then, the solutions of (4.7) *converge globally uniformly asymptotically to a certain ball centered at the origin.*

P r o o f. Let us take the Lyapunov function $V(t,x)$ satisfying conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and inequality (4.9). Now, we consider the upper right-hand derivative of $V(t,x)$ with respect the perturbed system (4.7). We have

$$D^+_{f+g}V(t,x) \leqslant D^+_f V(t,x) + K_2\|g(t,x)\|.$$

By using (4.8) and (4.9), we obtain

$$D^+_{f+g}V(t,x) \leqslant -\alpha(\|x\|) + K_2\big(\lambda(t,x)\alpha_0(\|x\|) + \xi(t,x)\big),$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. By using (4.10) and (4.11), we get

$$D^+_{f+g}V(t,x) \leqslant -\alpha(\|x\|) + K_2\big(\lambda_0\alpha_0(\|x\|) + \kappa\big),$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. By (4.12), we have

$$D^+_{f+g}V(t,x) \leqslant -(1-l)\alpha(\|x\|) + K_2\kappa.$$

Thus, the last inequality together with $(\mathcal{H}_2)$ yields

$$D^+_{f+g}V(t,x) \leqslant -(1-l)\alpha\Big(\frac{1}{K_1}V(t,x)\Big) + K_2\kappa.$$

So,

$$D^+_{f+g}V(t,x) \leqslant -\alpha_2\big(V(t,x)\big) + c,$$

where $\alpha_2(r) = (1-l)\alpha\big(\frac{1}{K_1}r\big) \in \mathcal{K}_\infty$ and $c = K_2\kappa$. By using the reasoning from the proof of Theorem 4.1, starting from (4.4) to the end of the proof of Theorem 4.1, we obtain that the solutions of (4.7) converge globally uniformly asymptotically to a certain ball centered at the origin, and the ball B_δ with $\delta = \alpha_2^{-1}(2K_2\kappa)$ is globally asymptotically stable for (4.7). □

Note that, if κ is small enough, then the radius of the ball also becomes small. Therefore, if the bound κ depends on a parameter ε with $\lim\limits_{\varepsilon\to 0}\kappa(\varepsilon) = 0$, then the state will approches to the origin exponentially when t tends to infinity and $\varepsilon \to 0$. This can be illustrated in the following examples.

Example 4.1. Consider the differential equation:

$$\dot{x} = -x + \varepsilon\frac{|x|}{|x|+1}\sin^2 t, \quad t \geqslant 0, \quad x \in \mathbb{R}, \tag{4.13}$$

with $\varepsilon > 0$. Equation (4.13) has the form (4.7), where $f(t,x) = -x$, $g(t,x) = \varepsilon\dfrac{|x|}{|x|+1}\sin^2 t$. Take $V(t,x) = |x|$. Then conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are fulfilled if $K_1 = K_2 = 1$. We have $D^+V(t,x) = -|x|$. Hence, inequality (4.9) holds for $\alpha(r) = r \in \mathcal{K}_\infty$. Next, inequality (4.8) holds for

$$\lambda(t,x) \equiv 0, \quad \alpha_0(r) = r, \quad \xi(t,x) = \varepsilon\sin^2 t.$$

Take $\kappa = \varepsilon$. Let $l \in (0,1)$ be arbitrary. Set $\lambda_0 := l$. Then inequalities (4.10), (4.11), (4.12) are fulfilled. We have $\alpha_2(r) = (1-l)r$, $c = \varepsilon$. By Theorem 4.2, solutions of (4.13) converge globally uniformly asymptotically to the ball B_δ with $\delta = \alpha_2^{-1}(2\varepsilon) = 2\varepsilon/(1-l)$

Example 4.2. Consider the differential equation:

$$\dot{x} = -\frac{1}{2}x^3 + \varepsilon\frac{1}{|x|+1}\sin^2 t, \quad t \geqslant 0, \quad x \in \mathbb{R}, \tag{4.14}$$

with $\varepsilon > 0$. Equation (4.14) has the form (4.7), where $f(t,x) = -x^3/2$, $g(t,x) = \varepsilon\frac{1}{|x|+1}\sin^2 t$. Take $V(t,x) = |x|$. Then conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are fulfilled if $K_1 = K_2 = 1$. We have $D^+V(t,x) = -|x|^3/2$. Hence, inequality (4.9) holds for $\alpha(r) = r^3/2 \in \mathcal{K}_\infty$. Next, inequality (4.8) holds for

$$\lambda(t,x) \equiv 0, \quad \alpha_0(r) = r^3/2, \quad \xi(t,x) = \varepsilon\sin^2 t.$$

Take $\kappa = \varepsilon$. Let $l \in (0,1)$ be arbitrary. Set $\lambda_0 := l$. Then inequalities (4.10), (4.11), (4.12) are fulfilled. We have $\alpha_2(r) = (1-l)r^3/2$, $c = \varepsilon$. By Theorem 4.2, solutions of (4.14) converge globally uniformly asymptotically to the ball B_δ with $\delta = \alpha_2^{-1}(2\varepsilon) = \big(4\varepsilon/(1-l)\big)^{1/3}$.

Example 4.3. Consider the differential equation:

$$\dot{x} = -x^3 + \varepsilon\sin t(x^3 + \cos x), \quad t \geqslant 0, \quad x \in \mathbb{R}, \tag{4.15}$$

with $\varepsilon > 0$. Equation (4.15) has the form (4.7), where $f(t,x) = -x^3$, $g(t,x) = \varepsilon\sin t(x^3+\cos x)$. Take $V(t,x) = |x|$. Then conditions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are fulfilled if $K_1 = K_2 = 1$. We have $D^+V(t,x) = -|x|^3$. Hence, inequality (4.9) holds for $\alpha(r) = r^3 \in \mathcal{K}_\infty$. Next, inequality (4.8) holds for

$$\lambda(t,x) = \varepsilon|\sin t|, \quad \alpha_0(r) = r^3, \quad \xi(t,x) = \varepsilon|\sin t\cos x|.$$

Set $\lambda_0 := \varepsilon$ and $\kappa := \varepsilon$. Then inequalities (4.10) and (4.11) are fulfilled. Suppose that $\varepsilon < 1$. Set $l := \varepsilon$. Then $l \in (0,1)$ and inequality (4.12) holds. We have $\alpha_2(r) = (1-\varepsilon)r^3$, $c = \varepsilon$. By Theorem 4.2, if $\varepsilon \in (0,1)$, then solutions of (4.15) converge globally uniformly asymptotically to the ball B_δ with $\delta = \alpha_2^{-1}(2\varepsilon) = \big(2\varepsilon/(1-\varepsilon)\big)^{1/3}$.

§ 5. Robustness with respect to time scaling

Consider, for some $\epsilon > 0$, the following system associated to (2.1):

$$\dot{y} = \epsilon f(\epsilon t, y), \tag{5.1}$$

where $f\colon \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and locally Lipschitz with respect to y such that $f(t,0) = 0$, $\forall t \geqslant 0$. For any $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$, we denote by $y(t,t_0,x_0)$, or simply by $y(t)$, the unique solution of (5.1) at time t_0 starting from the point x_0. We have, $\forall t \geqslant t_0 \geqslant 0$,

$$y(t,t_0,x_0) = x_0 + \int_{t_0}^{t} \epsilon f\big(\epsilon s, y(s,t_0,x_0)\big)\, ds.$$

Remark that, one has

$$y(t,t_0,x_0) = x(\epsilon t, \epsilon t_0, x_0), \quad \forall t \geqslant t_0 \geqslant 0.$$

Indeed,

$$\frac{d}{dt}y(t,t_0,x_0) = \frac{d}{dt}x(\epsilon t,\epsilon t_0,x_0) = \epsilon\frac{d}{d(\epsilon t)}x(\epsilon t,\epsilon t_0,x_0),$$
$$\frac{d}{dt}y(t,t_0,x_0) = \epsilon f\big(\epsilon t, y(t,t_0,x_0)\big) = \epsilon f\big(\epsilon t, x(\epsilon t,\epsilon t_0,x_0)\big),$$

and $y(t,t_0,x_0)\big|_{t=t_0} = x_0 = x(\epsilon t,\epsilon t_0,x_0)\big|_{t=t_0}$.

Theorem 5.1. *Suppose that there exists $r > 0$ such that the ball B_r is globally uniformly asymptotically stable with respect to the system (2.1). Then, for any $\epsilon > 0$, B_r is globally uniformly asymptotically stable with respect to the system (5.1) as well.*

P r o o f. Taking into account Definition 2.6, there exists a class $\mathcal{KL}$ function β such that

$$\|x(t,t_0,x_0)\| \leqslant \beta(\|x_0\|, t-t_0) + r, \quad \forall\, t \geqslant t_0 \geqslant 0, \quad \forall x_0 \in \mathbb{R}^n \setminus B_r.$$

Thus, using the fact that $y(t,t_0,x_0) = x(\epsilon t, \epsilon t_0, x_0)$, $\forall t \geqslant t_0 \geqslant 0$, which is the corresponding solution of (5.1), one gets

$$\|y(t,t_0,x_0)\| = \|x(\epsilon t, \epsilon t_0, x_0)\| \leqslant \beta\big(\|x_0\|, \epsilon(t-t_0)\big) + r. \qquad \square$$

Example 5.1. Let us consider the scaling system associated to the one given in Example 4.1

$$\dot{y} = -\epsilon y + \varepsilon \frac{|\epsilon y|}{|y|+1} \sin^2 \epsilon t, \quad x \in \mathbb{R}, \quad \epsilon > 0. \tag{5.2}$$

Since the solution $x(t,t_0,x_0)$ satisfies

$$\|x(t,t_0,x_0)\| \leqslant \beta(\|x_0\|, (t-t_0)) + 2\varepsilon/(1-l),$$

for certain $\mathcal{KL}$ function β, then, one has

$$\|y(t,t_0,x_0)\| \leqslant \beta(\|y_0\|, \epsilon(t-t_0)) + 2\varepsilon/(1-l).$$

It follows that, the ball B_δ with $\delta = 2\varepsilon/(1-l)$ is globally asymptotically stable with respect to the system (5.2).

Conclusion

In this paper, some new sufficient conditions for the asymptotic or exponential stability of a class of nonlinear time-varying differential equations have been presented by using Lyapunov functions that are not necessarily differentiable. The notions of stability in variation and Lipschitz stability have been discussed as well. Moreover, the global uniform asymptotic stability for perturbed nonautonomous systems by using Lyapunov approach has been studied. The present results have been applied to some examples.

Funding. The study of V. A. Zaitsev was funded by the Ministry of Science and Higher Education of the Russian Federation in in the framework of the state assignment, project FEWS-2024-0009.

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Received 19.10.2023

Accepted 27.05.2024

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Citation: M. A. Hammami, R. Hamlili, V. A. Zaitsev. On the stability in variation of non-autonomous differential equations with perturbations, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2024, vol. 34, issue 2, pp. 222–247.

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Об устойчивости в вариации неавтономных дифференциальных уравнений с возмущениями

Ключевые слова: неавтономные дифференциальные уравнения, возмущение, функции Ляпунова, асимптотическая устойчивость.

УДК 517.9

DOI: 10.35634/vm240204

В данной статье исследуется проблема устойчивости в вариации решений неавтономных дифференциальных уравнений. Представлены некоторые новые достаточные условия асимптотической или экспоненциальной устойчивости для некоторых классов нелинейных нестационарных дифференциальных уравнений, использующие функции Ляпунова, которые не обязательно являются гладкими. Предлагаемый подход для анализа устойчивости основан на определении границ, характеризующих асимптотическую сходимость решений к некоторому замкнутому множеству, содержащему начало координат. Кроме того, приведены некоторые иллюстративные примеры, демонстрирующие справедливость основных результатов.

Финансирование. Исследования В. А. Зайцева выполнены при поддержке Министерства науки и высшего образования РФ в рамках государственного задания, проект FEWS-2024-0009.

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https://doi.org/10.3390/math8050853

Поступила в редакцию 19.10.2023

Принята к публикации 27.05.2024

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Цитирование: М. А. Хаммами, Р. Хамлили, В. А. Зайцев. Об устойчивости в вариации неавтономных дифференциальных уравнений с возмущениями // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2024. Т. 34. Вып. 2. С. 222–247.