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ON SOLVING TERMINAL APPROACH AND EVASION PROBLEMS FOR LINEAR DISCRETE-TIME SYSTEMS UNDER STATE CONSTRAINTS

The paper is devoted to elaboration of polyhedral techniques for solving two control problems for linear discrete-time systems with uncertainties under state constraints, namely, the terminal approach problem and the terminal evasion one. Such problems arise in systems with two controls, where the aim of the first is to steer the trajectory onto a given terminal set at a given instant without violating the state constraints, the aim of the other is opposite. It is assumed that the terminal set is a parallelepiped, the controls are bounded by parallelotope-valued constraints, and the state constraints are given in the form of so-called zones. We present techniques for solving both problems basing on polyhedral (parallelotope-valued or parallelepiped-valued) tubes. The techniques for solving the approach problem were proposed by the author earlier, but here additional properties of them are investigated. In particular, for the case without state constraints, guaranteed estimates are found for the trajectory that ensure that it is inside the tube. Convenient sufficient conditions are given to guarantee the obtaining of nondegenerate cross-sections during the calculations. For the evasion problem, a common solution scheme is considered, and then polyhedral techniques are proposed. The whole parametric families of external and internal polyhedral estimates for the solvability tubes for both problems are presented and compared. An illustrative example is given.

Keywords: systems with uncertainties, control synthesis, approach problem, evasion problem, polyhedral methods, parallelotopes, parallelepipeds.

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Introduction

We consider linear discrete-time systems under state constraints with conflicting controls, where the aim of the first one is to steer the trajectory onto a given terminal set at a given instant without violating the state constraints; the aim of the other is opposite. We deal with two control problems under uncertainties: the approach problem and the evasion problem.

There are known approaches to solve problems of this kind based on construction of set-valued solvability tubes [1–4]. Since such tubes can be calculated exactly only in rare cases, various numerical methods have been developed, in particular using unions of a lot of points or polytopes with many vertices and faces [2–6] (here and below we mention as examples only some references from a lot of publications; the appropriate references can be also found in them). But the methods for obtaining the most accurate approximations of sets can require large calculations. Therefore a group of methods is based on estimating sets by domains of simple fixed shape such as ellipsoids [2, 3, 7–10] or parallelepipeds/parallelotopes [11–14].

In particular, constructive schemes to solve approach problems were developed through ellipsoidal techniques [2, 3, 8]. Polyhedral techniques using parallelotopes were developed to solve the approach problems [15], including the case with state constraints [16], and then for evasion problems [14, 17]. The devised polyhedral techniques were successfully applied to solve some aircraft control problems with disturbances [6, 14, 18].

In the paper, we present techniques to solve the terminal evasion problems for linear discrete-time systems under state constraints. The common solution scheme is considered basing on construction of solvability tubes. Then, under appropriate assumptions on the terminal set and on the sets that bound the controls and that define the state constraints, we present much more quick and simple for realization method basing on polyhedral tubes (the tubes with parallelepiped-valued or parallelotope-valued cross-sections). Control strategies are determined by these tubes via explicit formulas. For completeness of exposition and convenience of comparison of the used constructions we recall similar results concerning the approach problems. The whole parametric families of external and internal polyhedral estimates for the solvability tubes for both problems are presented and compared. Here each polyhedral tube can be calculated independently of the others by recurrence relations that involve elementary polyhedral estimates for results of operations with sets. New helpful properties of elementary estimates are presented. The method for solving the approach problem developed by the author earlier is investigated here in more detail. In particular, for the case without state constraints, guaranteed estimates are found for the trajectory that ensure that it is inside the tube. Convenient sufficient conditions are given to guarantee the obtaining of nondegenerate cross-sections during the calculations. An illustrative example is given.

Among other interesting results connected with solving control problems under uncertainties and evasion problems we also note [19,20].

Let us introduce the notation used below. Let $\mathbb{R}^n$ stand for the n-dimensional vector space; $(x,y) = x^\top y$ be the scalar product for $x,y \in \mathbb{R}^n$; $\top$ denote the transposition symbol; $\|x\|_\infty = \max_{1\le i\le n}|x_i|$ stand for the maximum norm of the vector $x = (x_1,\dots,x_n)^\top \in \mathbb{R}^n$; $\mathrm{e}^i = (0,\dots,0,1,0,\,\dots,0)^\top$ be the unit vector oriented along the axis x_i (the unit stands at position i); $\mathrm{e} = (1,1,\,\dots,1)^\top$. Inequalities $\le,<,\ge,>$ for vectors are understood componentwise. Denote by $\mathbb{R}^{m\times n}$ the space of real $m{\times}n$-matrices $A = \{a_i^j\} = \{a^j\}$ with elements a_i^j and columns a^j, where the superscript is used to enumerate the columns, and the subscript is used to enumerate vector components. Let 0 be the zero matrix (vector) and I be the identity matrix. By $\operatorname{diag}\pi$, $\operatorname{diag}\{\pi_i\}$ we denote the diagonal matrix A with diagonal elements $a_i^i = \pi_i$, where π_i stand for the components of the vector π. A matrix $A \in \mathbb{R}^{n\times n}$ is called nonsingular if $\det A \ne 0$. Set $\operatorname{Abs} A = \{|a_i^j|\}$ for $A = \{a_i^j\} \in \mathbb{R}^{m\times n}$. Let $\|\Gamma\|_\infty = \max\limits_{1\le i\le m}\sum\limits_{j=1}^{n}|\gamma_i^j|$ denote the matrix norm for $\Gamma = \{\gamma_i^j\} \in \mathbb{R}^{m\times n}$, and $\mathcal{G}^{m\times n} = \{\Gamma \in \mathbb{R}^{m\times n} \mid \|\Gamma\|_\infty \le 1\}$. The set of interior points of the set $\mathcal{X} \subset \mathbb{R}^n$ is denoted by $\operatorname{int}\mathcal{X}$. We deal with following operations with sets: *Minkowski's sum* (or *geometrical sum*) $\mathcal{X}^1 + \mathcal{X}^2 = \{y \mid y = x^1 + x^2, x^k \in \mathcal{X}^k\}$, *Minkowski's* (or *geometrical*) *difference* $\mathcal{X}^1 \dot{-} \mathcal{X}^2 = \{y \mid y + \mathcal{X}^2 \subseteq \mathcal{X}^1\}$, affine transformation, intersection, and *complement* $\mathbb{R}^n \setminus \mathcal{X} = \{x \in \mathbb{R}^n \mid x \notin \mathcal{X}\}$. Let $\rho(l\,|\,\mathcal{Q}) = \sup\{l^\top x \mid x \in \mathcal{Q}\}$ be the support function of $\mathcal{Q} \subset \mathbb{R}^n$. The symbol $\operatorname{sign} z$ stands for the function with values -1, 0, 1 for $z<0$, $z=0$, $z>0$ respectively. To be short, we write $k=1,\dots,N$ instead of $k=1,2,\dots,N$.

§ 1. Statement of the problems

We deal with the linear discrete-time system ($x \in \mathbb{R}^n$)

$$x[k] = A[k]x[k-1] + B[k]u[k] + C[k]v[k], \quad k = 1,\dots,N, \tag{1.1}$$

with the given terminal set $\mathcal{M} \subset \mathbb{R}^n$ and controls $u[k] \in \mathbb{R}^{n_u}$ and $v[k] \in \mathbb{R}^{n_v}$ subjected to

$$u[k] \in \mathcal{R}[k], \quad v[k] \in \mathcal{Q}[k], \quad k = 1,\dots,N, \tag{1.2}$$

where $\mathcal{R}[k]$, $\mathcal{Q}[k]$ are given sets. We call the functions $u[\cdot]$ and $v[\cdot]$ satisfying (1.2) *admissible*. The system may be complicated by state constraints determined by the given sets $\mathcal{Y}[k] \subset \mathbb{R}^n$. The matrices $A[k] \in \mathbb{R}^{n\times n}$, $B[k] \in \mathbb{R}^{n\times n_u}$, $C[k] \in \mathbb{R}^{n\times n_v}$ are given.

Throughout the paper we accept the following assumption and don't mention it anymore.

Assumption 1.1. All matrices $A[k]$ are nonsingular; $\mathcal{R}[k]$, $\mathcal{Q}[k]$, and $\mathcal{M}$ are convex compact sets; $\mathcal{Y}[k]$ are convex closed sets.

The controls u and v have different aims, which may be called approach and evasion. The aim of u is to ensure $x[N] \in \mathcal{M}$ and

$$x[k] \in \mathcal{Y}[k] \subset \mathbb{R}^n, \quad k = 0, \dots, N-1. \tag{1.3}$$

The aim of v is to ensure either $x[N] \notin \mathcal{M}$ or violation of (1.3) for some $k \in \{0, \dots, N-1\}$.

There are well known approaches [1, 2], which allow to solve such problems using special tubes $\mathcal{W}[\cdot]$ and $\hat{\mathcal{W}}[\cdot]$ (or $\check{\mathcal{W}}[\cdot]$), i. e., multi-valued functions with set-valued cross-sections $\mathcal{W}[k]$ and $\hat{\mathcal{W}}[k]$ (or $\check{\mathcal{W}}[k]$), $k = 0, \dots, N$. Let us formulate the corresponding problems.

Problem 1 (approach problem). Find a *solvability tube* $\mathcal{W}[\cdot]$ satisfying $\mathcal{W}[N] = \mathcal{M}$ and $\mathcal{W}[k] \subseteq \mathcal{Y}[k]$, $k = 0, \dots, N$, and a feedback control strategy $u = u[k, x]$ with $u[k, x] \in \mathcal{R}[k]$ such that each solution $x[\cdot]$ to

$$x[k] = A[k]x[k-1] + B[k]u[k, x[k-1]] + C[k]v[k], \quad k = 1, \dots, N, \tag{1.4}$$

that starts from any $x[0] = x^0 \in \mathcal{W}[0]$ would be inside $\mathcal{W}[\cdot]$ ($x[k] \in \mathcal{W}[k]$, $k = 1, \dots, N$) whatever are admissible functions $v[\cdot]$ subjected to (1.2).

Problem 2 (evasion problem). Find a tube $\hat{\mathcal{W}}[\cdot]$ (or $\check{\mathcal{W}}[\cdot]$) with $\hat{\mathcal{W}}[N]{=}\mathcal{M}$ ($\check{\mathcal{W}}[N]{=}\mathbb{R}^n \setminus \mathcal{M}$) and a feedback control strategy $v = v[k, x]$ with $v[k, x] \in \mathcal{Q}[k]$ such that each solution $x[\cdot]$ to

$$x[k] = A[k]x[k-1] + B[k]u[k] + C[k]v[k, x[k-1]], \quad k = 1, \dots, N, \tag{1.5}$$

that starts at $x[0] = x^0 \notin \hat{\mathcal{W}}[0]$ ($x^0 \in \check{\mathcal{W}}[0]$) would satisfy one of two following conditions: either $x[k] \notin \hat{\mathcal{W}}[k]$ ($x[k] \in \check{\mathcal{W}}[k]$), $k = 1, \dots, N$, or $x[k] \notin \mathcal{Y}[k]$ for some $k \in \{0, \dots, N-1\}$ whatever are admissible functions $v[\cdot]$ subjected to (1.2).

The above tubes are not uniquely determined by the above conditions. Below, we use the notation $\mathcal{W}[\cdot]$, $\hat{\mathcal{W}}[\cdot]$, $\check{\mathcal{W}}[\cdot]$ for the tubes that are maximal, minimal, maximal (with respect to inclusion) respectively; we have $\check{\mathcal{W}}[k] = \mathbb{R}^n \setminus \hat{\mathcal{W}}[k]$. Exact calculation of such tubes is, as a rule, quite cumbersome. Therefore let us set, similarly to [15–17, 21], polyhedral approach and evasion problems whose solutions are based on construction of tubes $\mathcal{P}^-[\cdot]$ and $\hat{\mathcal{P}}^+[\cdot]$ with parallelotope-valued or parallelepiped-valued cross-sections. We call them *polyhedral tubes*.

Assumption 1.2. The terminal set $\mathcal{M}$ is a parallelepiped $\mathcal{M} = \mathcal{P}(p_{\mathrm{f}}, P_{\mathrm{f}}, \pi_{\mathrm{f}}) = \mathcal{P}[p_{\mathrm{f}}, \bar{P}_{\mathrm{f}}]$; the constraining sets $\mathcal{R}[k]$ and $\mathcal{Q}[k]$ are parallelotopes $\mathcal{R}[k] = \mathcal{P}[r[k], \bar{\mathcal{R}}[k]]$, $\mathcal{Q}[k] = \mathcal{P}[q[k], \bar{Q}[k]]$ with $\bar{R}[k] \in \mathbb{R}^{n_u \times n_u}$, $\bar{Q}[k] \in \mathbb{R}^{n_v \times n_v}$; $\mathcal{Y}[k]$ are either zones or $\mathcal{Y}[k] = \mathbb{R}^n$ (the equality $\mathcal{Y}[k] = \mathbb{R}^n$ means that there are no state constraints at time k).

By a *parallelepiped* $\mathcal{P}(p, P, \pi) \subset \mathbb{R}^n$ we call a set of the form $\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \in \mathbb{R}^n \mid x = p + P \operatorname{diag} \pi \cdot \zeta,\ \|\zeta\|_\infty \le 1\}$, where $p \in \mathbb{R}^n$; $P = \{p^i\} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix with $\|p^i\|_2 = 1$; $\pi \in \mathbb{R}^n$, $\pi \ge 0$. To simplify formulas the conditions $\|p^i\|_2 = 1$ for the Euclidean norm can be omitted. We say that p is the parallelepiped center, P is the orientation matrix. A parallelepiped is said to be *nondegenerate* if $\pi > 0$.

By a *parallelotope* $\mathcal{P}[p, \bar{P}] \subset \mathbb{R}^n$ we call a set of the form $\mathcal{P} = \mathcal{P}[p, \bar{P}] = \{x \mid x = p + \bar{P}\,\zeta,\ \|\zeta\|_\infty \le 1\}$. Here $p \in \mathbb{R}^n$, $\bar{P} = \{\bar{p}^i\} \in \mathbb{R}^{n \times m}$, where $m \le n$. A parallelotope $\mathcal{P}$ is said to be *nondegenerate* if $m = n$ and $\det \bar{P} \ne 0$.

By a *zone* (or m-*zone*) $\mathcal{S} = \mathcal{S}(c, S, \sigma, m) \subset \mathbb{R}^n$ we call a set that is an intersection of $m \le n$ *strips* Σ^i: $\mathcal{S} = \mathcal{S}(c, S, \sigma, m) = \bigcap\limits_{i=1}^{m} \Sigma^i$, $\Sigma^i = \Sigma(c_i, s^i, \sigma_i) = \{x \mid |(x, s^i) - c_i| \le \sigma_i\}$. Here $c \in \mathbb{R}^m$; $S = \{s^i\} \in \mathbb{R}^{n\times m}$, where vectors s^i are linearly independent; $\sigma \in \mathbb{R}^m$, $\sigma \ge 0$.

Any parallelepiped $\mathcal{P}(p, P, \pi)$ is, in fact, a parallelotope $\mathcal{P}[p, \bar{P}]$, where $\bar{P} = P\,\mathrm{diag}\,\pi$. Any nondegenerate parallelotope $\mathcal{P} = \mathcal{P}[p, \bar{P}]$ is a parallelepiped $\mathcal{P} = \mathcal{P}(p, \bar{P}, \mathrm{e})$, where $\mathrm{e} = (1, \ldots, 1)^\top$. Any parallelepiped is a zone, and vice versa for $m = n$ (see [12, 13]).

Sometimes we also accept the following assumption.

Assumption 1.3. The terminal set $\mathcal{M}$ is a nondegenerate parallelotope, i. e., $\det \bar{P}_{\mathrm{f}} \neq 0$.

Problem 3. For system (1.1)–(1.3) under Assumptions 1.2, 1.3 find a polyhedral tube $\mathcal{P}^-[\cdot]$ with $\mathcal{P}^-[N] = \mathcal{M}$ and $\mathcal{P}^-[k] \subseteq \mathcal{Y}[k]$, $k = 0, \ldots, N-1$, and find a corresponding feedback control strategy $u = u[k, x]$ satisfying $u[k, x] \in \mathcal{R}[k]$, $k = 1, \ldots, N$, such that every solution $x[\cdot]$ to (1.4) that starts at $x[0] = x^0 \in \mathcal{P}^-[0]$ would be inside $\mathcal{P}^-[\cdot]$ ($x[k] \in \mathcal{P}^-[k]$, $k = 1, \ldots, N$), whatever is an admissible $v[\cdot]$. Moreover, describe a family of such tubes $\mathcal{P}^-[\cdot]$.

Problem 4. For system (1.1), (1.2) with the given sets $\mathcal{Y}[k]$ under Assumption 1.2 find a polyhedral tube $\hat{\mathcal{P}}^+[\cdot]$ with $\hat{\mathcal{P}}^+[N] \supseteq \mathcal{M}$ and find a corresponding feedback control strategy $v = v[k, x]$ satisfying $v[k, x] \in \mathcal{Q}[k]$, $k = 1, \ldots, N$, such that every solution $x[\cdot]$ to (1.5) that starts at $x[0]{=}x^0 \notin \hat{\mathcal{P}}^+[0]$ would satisfy one of two following conditions (either (1.6) or (1.7)):

$$x[k] \notin \hat{\mathcal{P}}^+[k], \quad k = 0, \ldots, N, \tag{1.6}$$

$$x[k] \notin \mathcal{Y}[k] \quad \text{for some} \;\; k \in \{0, \ldots, N-1\} \tag{1.7}$$

whatever is an admissible $u[\cdot]$, i. e., either the trajectory would be outside $\hat{\mathcal{P}}^+[\cdot]$ and therefore $x[N] \notin \mathcal{M}$ or the constraints (1.3) would be violated. Describe a family of such tubes $\hat{\mathcal{P}}^+[\cdot]$.

Thus the set $\mathcal{W}^-[0] = \bigcup \mathcal{P}^-[0]$, where the union is taken over the tubes $\mathcal{P}^-[\cdot]$ from Problem 3, is a subset of the set $\mathcal{W}[0]$ from Problem 1 of all initial points x^0 for which the aim of the control u is achievable.

Similarly, the set $\hat{\mathcal{W}}^+[0] = \bigcap \hat{\mathcal{P}}^+[0]$, where the intersection is taken over the tubes $\hat{\mathcal{P}}^+[\cdot]$ from Problem 4, serves as an external estimate for the set $\hat{\mathcal{W}}[0]$ from Problem 2 such that the set $\check{\mathcal{W}}[0] = \mathbb{R}^n \setminus \hat{\mathcal{W}}[0]$ is the set of all x^0 for which the aim of the control v is achievable.

Thus we have $\bigcup \mathcal{P}^-[0] \subseteq \mathcal{W}[0] \subseteq \hat{\mathcal{W}}[0] \subseteq \bigcap \hat{\mathcal{P}}^+[0]$.

Below we also describe other families of external estimates $\mathcal{P}^+[\cdot]$ for $\mathcal{W}[\cdot]$ and internal estimates $\hat{\mathcal{P}}^-[\cdot]$ for $\hat{\mathcal{W}}[\cdot]$ so that $\mathcal{W}[0] \subseteq \bigcap \mathcal{P}^+[0]$ and $\bigcup \hat{\mathcal{P}}^-[0] \subseteq \hat{\mathcal{W}}[0]$.

§ 2. Elementary polyhedral estimates for operations with sets

Solutions to Problems 3 and 4 can be found using elementary polyhedral estimates for operations with sets. For convenience let us recall the main constructions to be used.

The Minkowski difference $\mathcal{Q} = \mathcal{P}^1 \dot{-} \mathcal{P}^2$ of the parallelepiped $\mathcal{P}^1 = \mathcal{P}(p^1, P^1, \pi^1) = \mathcal{P}[p^1, \bar{P}^1]$ and the parallelotope $\mathcal{P}^2 = \mathcal{P}[p^2, \bar{P}^2]$ is either a parallelepiped or an empty set. Namely, set $\pi^{\mathrm{dif}} = \pi^1 - (\mathrm{Abs}\,((P^1)^{-1}\bar{P}^2))\,\mathrm{e}$. Then, similarly to [12, Lemma 3.15], we have $\mathcal{Q} = \mathcal{P}(p^1 - p^2, P^1, \pi^{\mathrm{dif}})$ if $\pi^{\mathrm{dif}} \ge 0$; otherwise $\mathcal{Q} = \varnothing$. In terms of parallelotopes, set $\pi^* = \mathrm{e} - (\mathrm{Abs}\,((\bar{P}^1)^{-1}\bar{P}^2))\,\mathrm{e}$. Then $\mathcal{Q} = \mathcal{P}[p^1 - p^2, \bar{P}^1 \mathrm{diag}\,\pi^*]$ if $\pi^* \ge 0$; otherwise $\mathcal{Q} = \varnothing$.

The set $\mathcal{P}^-$ ($\mathcal{P}^+$) is called *internal (external) estimate for* $\mathcal{Q} \subset \mathbb{R}^n$ if $\mathcal{P}^- \subseteq \mathcal{Q}$ ($\mathcal{P}^+ \supseteq \mathcal{Q}$). The so called *touching external estimate* $\boldsymbol{P}_V^+(\mathcal{Q})$ for the set $\mathcal{Q}$ with the orientation matrix V is defined by the relations $\rho(\pm(V^{-1})^\top \mathrm{e}^i | \boldsymbol{P}_V^+(\mathcal{Q})) = \rho(\pm(V^{-1})^\top \mathrm{e}^i | \mathcal{Q})$, $i = 1, \ldots, n$, and can be found using values of the support function for $\mathcal{Q}$ [12, 13]. We define $\boldsymbol{P}_V^+(\varnothing) = \varnothing$.

The touching external estimate $\mathcal{P}^+ = \boldsymbol{P}_V^+(\mathcal{Q})$ for the Minkowski sum $\mathcal{Q} = \mathcal{P}^1 + \mathcal{P}^2$ of the parallelepiped and the parallelotope is calculated [12] by the formula

$$\boldsymbol{P}_V^+(\mathcal{P}^1 + \mathcal{P}^2) = \mathcal{P}(p^1 + p^1, V, (\mathrm{Abs}\,(V^{-1}P^1))\,\pi^1 + (\mathrm{Abs}\,(V^{-1}\bar{P}^2))\,\mathrm{e}) =$$
$$= \mathcal{P}[p^1 + p^1, V \cdot \mathrm{diag}\,(\sum_{k=1}^{2}(\mathrm{Abs}\,(V^{-1}\bar{P}^k)))\,\mathrm{e}].$$

Recall the formulas for operating with empty sets to be used below: $\mathcal{X} + \varnothing = \varnothing$, $\varnothing \dot{-} \mathcal{X} = \varnothing$, $\mathcal{X} \dot{-} \varnothing = \mathbb{R}^n$, $\mathcal{X} \cap \varnothing = \varnothing$, $\mathcal{X} \cup \varnothing = \mathcal{X}$, $\mathbb{R}^n \setminus \varnothing = \mathbb{R}^n$, $A \cdot \varnothing = \varnothing$.

Consider the operations of the type $\mathcal{X} = (\mathcal{P}^1 \dot{-} \mathcal{P}^3) + \mathcal{P}^2$ and $\hat{\mathcal{X}} = (\mathcal{P}^1 + \mathcal{P}^2) \dot{-} \mathcal{P}^3$, called in [8] *geometric difference-sum* and *geometric sum-difference*, and external estimates for them: $\mathcal{X} \subseteq \mathcal{P}^+$, $\hat{\mathcal{X}} \subseteq \hat{\mathcal{P}}^+$. Note that we have $\mathcal{X} \subseteq \hat{\mathcal{X}}$ due to [22, (3.1.13)].

Lemma 2.1. *Let* $\mathcal{P}^1 = \mathcal{P}(p^1, P^1, \pi^1)$, $\mathcal{P}^k = \mathcal{P}[p^k, \bar{P}^k]$, $k = 2, 3$, *and* $\mathcal{X} = (\mathcal{P}^1 \dot{-} \mathcal{P}^3) + \mathcal{P}^2$, $\hat{\mathcal{X}} = (\mathcal{P}^1 + \mathcal{P}^2) \dot{-} \mathcal{P}^3$. *For arbitrary nonsingular matrix* V, *let*

$$\mathcal{P}^+ := \boldsymbol{P}_V^+(\mathcal{X}) = \boldsymbol{P}_V^+((\mathcal{P}^1 \dot{-} \mathcal{P}^3) + \mathcal{P}^2), \qquad \hat{\mathcal{P}}^+ := \boldsymbol{P}_V^+(\mathcal{P}^1 + \mathcal{P}^2) \dot{-} \mathcal{P}^3,$$

and

$$\nu^0 = \pi^1 - (\mathrm{Abs}\,((P^1)^{-1}\bar{P}^3))\,\mathrm{e}, \quad \nu = (\mathrm{Abs}\,(V^{-1}P^1))\,\nu^0 + (\mathrm{Abs}\,(V^{-1}\bar{P}^2))\,\mathrm{e};$$
$$\hat{\nu} = (\mathrm{Abs}\,(V^{-1}P^1))\,\pi^1 + (\mathrm{Abs}\,(V^{-1}\bar{P}^2))\,\mathrm{e} - (\mathrm{Abs}\,(V^{-1}\bar{P}^3))\,\mathrm{e}.$$

We have $\mathcal{X} \subseteq \mathcal{P}^+$ *and obtain* $\mathcal{P}^+ = \mathcal{P}(p^1 + p^2 - p^3, V, \nu)$ *for* $\nu^0 \geq 0$, *otherwise* $\mathcal{X} = \mathcal{P}^+ = \varnothing$. *We have* $\hat{\mathcal{X}} \subseteq \boldsymbol{P}_V^+(\hat{\mathcal{X}}) \subseteq \hat{\mathcal{P}}^+$ *and obtain* $\hat{\mathcal{P}}^+ = \mathcal{P}(p^1 + p^2 - p^3, V, \hat{\nu})$ *for* $\hat{\nu} \geq 0$, *otherwise* $\hat{\mathcal{X}} = \hat{\mathcal{P}}^+ = \varnothing$. *For the case* $\nu^0 \geq 0$, *under special choice* $V = P^1$, *we obtain* $\hat{\mathcal{P}}^+ = \mathcal{P}^+$.

For two parallelepipeds $\mathcal{P}^{1,1}$ *and* $\mathcal{P}^{1,2}$ *such that* $\mathcal{P}^{1,1} \subseteq \mathcal{P}^{1,2}$, *the following inclusion holds for the estimates* $\mathcal{P}^{+,1} = \boldsymbol{P}_V^+((\mathcal{P}^{1,1} \dot{-} \mathcal{P}^3) + \mathcal{P}^2)$ *and* $\hat{\mathcal{P}}^{+,2} = \boldsymbol{P}_V^+(\mathcal{P}^{1,2} + \mathcal{P}^2) \dot{-} \mathcal{P}^3$ *corresponding to one the same orientation matrix* V: $\mathcal{P}^{+,1} \subseteq \hat{\mathcal{P}}^{+,2}$.

P r o o f. The statements concerning $\mathcal{X}$, $\mathcal{P}^+$ and formulas for $\hat{\mathcal{P}}^+$ follow from the above. The inclusion $\hat{\mathcal{X}} \subseteq \hat{\mathcal{P}}^+$ is clear, and $\boldsymbol{P}_V^+(\hat{\mathcal{X}}) \subseteq \hat{\mathcal{P}}^+$ follows from [12, Lemma 3.4]. The formulas for ν and $\hat{\nu}$ under $V = P^1$ give $\hat{\mathcal{P}}^+ = \mathcal{P}^+$. The inclusion $\mathcal{P}^{+,1} \subseteq \hat{\mathcal{P}}^{+,2}$ can be obtained using inclusion monotonicity for the involved set operations, the relation $(\mathcal{P}^{1,2} \dot{-} \mathcal{P}^3) + \mathcal{P}^2 \subseteq (\mathcal{P}^{1,2} + \mathcal{P}^2) \dot{-} \mathcal{P}^3$ (see [22, (3.1.13)]), and $\boldsymbol{P}_V^+((\mathcal{P}^{1,2} + \mathcal{P}^2) \dot{-} \mathcal{P}^3) \subseteq \hat{\mathcal{P}}^{+,2}$, which is similar to $\boldsymbol{P}_V^+(\hat{\mathcal{X}}) \subseteq \hat{\mathcal{P}}^+$. □

An internal parallelotope-valued estimate for the sum $\mathcal{Q} = \mathcal{P}^1 + \mathcal{P}^2$ of two parallelotopes $\mathcal{P}^k = \mathcal{P}[p^k, \bar{P}^k]$ with $\bar{P}^1 \in \mathbb{R}^{n \times n}$, $\bar{P}^2 \in \mathbb{R}^{n \times r}$ can be found [13, Lemma 3.1] in the form $\mathcal{P}^- = \boldsymbol{P}^-_{\Gamma^1,\Gamma^2}(\mathcal{P}^1 + \mathcal{P}^2) = \mathcal{P}[p^1 + p^2, \bar{P}^1\Gamma^1 + \bar{P}^2\Gamma^2]$, where $\Gamma^1 \in \mathcal{G}^{n \times n}$, $\Gamma^2 \in \mathcal{G}^{r \times n}$.

The above matrices V, Γ^1, Γ^2 serve as parameters generating families of the estimates.

Originally, it is unclear how to choose Γ^1, Γ^2 to single out nondegenerate internal estimates $\mathcal{P}^- = \boldsymbol{P}^-_{\Gamma^1,\Gamma^2}(\mathcal{P}^1 + \mathcal{P}^2)$ for $\mathcal{Q} = \mathcal{P}^1 + \mathcal{P}^2$ if $\mathcal{P}^1$ is a nondegenerate parallelotope.

Proposition 2.1. *Let* $\mathcal{P}^k = \mathcal{P}[p^k, \bar{P}^k]$, $k = 1, 2$, $\bar{P}^1 \in \mathbb{R}^{n \times n}$, $\bar{P}^2 \in \mathbb{R}^{n \times r}$, *and* $\det \bar{P}^1 \neq 0$. *Let* $\mathcal{P}^- = \mathcal{P}[p^-, \bar{P}^-] = \boldsymbol{P}^-_{I,\Gamma}(\mathcal{P}^1 + \mathcal{P}^2)$ *be internal estimate for* $\mathcal{P}^1 + \mathcal{P}^2$ *determined by arbitrary* $\Gamma \in \mathcal{G}^{r \times n}$. *Set* $\gamma = (\mathrm{Abs}\,((\bar{P}^1)^{-1}\bar{P}^2))\,\mathrm{e}$. *A sufficient condition for* $\mathcal{P}^-$ *to be a nondegenerate parallelotope (parallelepiped) for any* $\Gamma \in \mathcal{G}^{r \times n}$ *is* $\gamma < \mathrm{e}$. *Under this condition, we have*

$$|\det \bar{P}^-| \geq |\det \bar{P}^1| \cdot \prod_{i=1}^{n}(1 - \gamma_i).$$

Also, if $\det \bar{P}^1 \neq 0$, *then for* $\mathcal{P}^1 \dot{-} \mathcal{P}^2$ *the same condition* $\gamma < \mathrm{e}$ *guarantees nondegeneracy of the parallelotope* $\mathcal{P}^1 \dot{-} \mathcal{P}^2 = \mathcal{P}[p^1 - p^2, \bar{P}^1 \mathrm{diag}\,(\mathrm{e} - \gamma)]$.

P r o o f. We have $\bar{P}^{-}{=}\bar{P}^{1}{+}\bar{P}^{2}\Gamma{=}\bar{P}^{1}(I{+}(\bar{P}^{1})^{-1}\bar{P}^{2}\Gamma)$. Therefore, to obtain the inequality for $\det\bar{P}^{-}$, it is sufficient to apply [17, Lemma 1]. Other statements are clear from the above. $\square$

Estimates for $\mathcal{P}\bigcap\mathcal{S}$, where $\mathcal{P}$ is a parallelepiped and $\mathcal{S}$ is a zone, can be constructed by different ways. Let us mention only some of them.

Below, for external estimates of the intersection of a parallelepiped $\mathcal{P}$ and a set $\mathcal{Q}$, we will use the notation $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{Q})$ for any external (not necessarily touching) estimate for $\mathcal{P}\bigcap\mathcal{Q}$ with the orientation matrix V. We only suppose that $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{Q})\supseteq\mathcal{P}\bigcap\mathcal{Q}$ and we put $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{Q})=\mathcal{P}$ if $\mathcal{P}\subseteq\mathcal{Q}$, and also we put $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{Q})=\varnothing$ if it is already known that $\mathcal{P}\bigcap\mathcal{Q}=\varnothing$. We call V admissible if there are known formulas for constructing $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{Q})$ under V. Some ways for constructing such estimates are indicated below.

So, for example, for the intersection $\mathcal{Q}=\mathcal{P}^{1}\bigcap\mathcal{P}^{2}$ of two parallelepipeds $\mathcal{P}^{k}{=}\mathcal{P}(p^{k},P^{k},\pi^{k})$ one can find parallelepiped-valued estimates $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{Q})$ with arbitrary V using explicit formulas [12, Lemma 3.14]: $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}^{1}\bigcap\mathcal{P}^{2})=\boldsymbol{P}^{+}_{V}(\mathcal{P}^{1})\bigcap\boldsymbol{P}^{+}_{V}(\mathcal{P}^{2})$. Note two heuristic ways to choose V here: put either $V=P^{1}$ (choice I) or $V\in\operatorname{Argmin}_{V\in\{P^{1},P^{2}\}}\operatorname{vol}\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}^{1}\bigcap\mathcal{P}^{2})$ (choice II).

For the intersection $\mathcal{Q}=\mathcal{P}\bigcap\Sigma$ of a parallelepiped and a strip, one can find touching estimates $\boldsymbol{P}^{+}_{V}(\mathcal{P}\bigcap\Sigma)$ using explicit formulas for $n+1$ specific orientation matrices $V\in\{P,P^{1},\dots,P^{n}\}$ that appear due to [11] (such estimates are obtained by neglecting one of $n+1$ so called tight strips, which form $\mathcal{P}\bigcap\Sigma$; P stands for the orientation matrix of $\mathcal{P}$).

Let $\mathcal{S}=\bigcap\limits_{i=1}^{m}\Sigma^{i}$ be an intersection of several strips. For some orientation matrices V the estimates $\tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{S})$ can be found by explicit formulas sequentially via m steps on the base of the mentioned estimates:

$$\mathcal{P}^{+,0}=\mathcal{P};\quad \mathcal{P}^{+,i}=\boldsymbol{P}^{+}_{P^{+,i}}(\mathcal{P}^{+,i-1}\textstyle\bigcap\Sigma^{i}),\ i=1,\dots,m;\quad \tilde{\boldsymbol{P}}^{+}_{V}(\mathcal{P}\bigcap\mathcal{S})=\mathcal{P}^{+,m},\tag{2.1}$$

where $V=P^{+,m}$ is equal to the orientation matrix of $\mathcal{P}^{+,m}$. To choose the orientation matrix $P^{+,i}$ on the step i, one can, for example, either set $P^{+,i}=P^{+,i-1}$ (we call these Choice I) or use the arguments of local volume optimality [11] (we call these Choice III).

Some ways of constructing internal estimates $\boldsymbol{P}^{-}_{v,V}(\mathcal{Q})$ with centers $v\in\operatorname{int}\mathcal{Q}$ and arbitrary orientation matrices V for $\mathcal{Q}=\mathcal{P}\bigcap\mathcal{S}$, $\mathcal{S}=\bigcap\limits_{i=1}^{m}\Sigma^{i}$, can be found in [13, Sec. 3], [16, Sec. 2].

To check whether a point x belongs to the nondegenerate parallellotope $\mathcal{P}[p,\bar{P}]$ or to the parallelepiped $\mathcal{P}(p,P,\pi)$ it is useful to use relative coordinates $\zeta{=}\bar{P}^{-1}(x{-}p)$, $\xi{=}P^{-1}(x{-}p)$.

Lemma 2.2 (see [17], [21, Lemma 1]). *Let $x\in\mathbb{R}^{n}$. Given $\mathcal{P}=\mathcal{P}[p,\bar{P}]$, $\det\bar{P}\neq0$, let $\zeta=\bar{P}^{-1}(x-p)$. Then $x\in\mathcal{P}$ iff $\operatorname{Abs}\zeta\leq\mathrm{e}$ and $x\notin\mathcal{P}$ iff $|\zeta_{i_*}|>1$ for some $i_*\in\{1,\dots,n\}$, i. e., $\|\zeta\|_\infty>1$. Given $\mathcal{P}=\mathcal{P}(p,P,\pi)$, let $\xi=P^{-1}(x-p)$. Then $x\notin\mathcal{P}$ iff $|\xi_{i_*}|>\pi_{i_*}$ for some $i_*\in\{1,\dots,n\}$.*

§ 3. Solving approach problems 1 and 3

The solution to Problem 1 is known. To remind it, consider the recurrence relations

$$\begin{gathered}\mathcal{W}[k-1]=A[k]^{-1}((\mathcal{W}[k]\dot{-}C[k]\mathcal{Q}[k])+(-B[k]\mathcal{R}[k]))\textstyle\bigcap\mathcal{Y}[k-1],\\ k=N,\dots,1,\quad \mathcal{W}[N]=\mathcal{M}.\end{gathered}\tag{3.1}$$

Theorem 3.1 (see, for example, [16]). *Problem 1 is solvable on the time interval $[0,N]$ if system (3.1) has a solution such that $\mathcal{W}[k]\neq\varnothing$, $k=N,\dots,0$. Then this tube $\mathcal{W}[\cdot]$ from (3.1) is a maximal solvability tube and it gives a solution to Problem 1 with any control strategy $u[\cdot,\cdot]$ satisfying $u[k,x]\in\mathcal{U}[k,x]=\mathcal{R}[k]\bigcap\{u\mid B[k]u\in(\mathcal{W}[k]\dot{-}C[k]\mathcal{Q}[k])-A[k]x\}$, where we have $\mathcal{U}[k,x]\neq\varnothing$ for any $x\in\mathcal{W}[k-1]$.*

The solution to Problem 3 can be, in fact, extracted from [16, Theorem 3.1]. Consider parallelotope-valued tubes $\mathcal{P}^-[\cdot] = \mathcal{P}[p^-[\cdot], \bar{P}^-[\cdot]]$ that satisfy the following relations:

$$\begin{gathered}
\mathcal{P}^{0-}[k-1] = \mathcal{P}^-[k] \dot{-} C[k]\mathcal{Q}[k], \quad k = N, \ldots, 1, \\
\mathcal{P}^{1-}[k-1] = A[k]^{-1}\boldsymbol{P}^-_{I,\Gamma[k]}(\mathcal{P}^{0-}[k-1] + (-B[k]\mathcal{R}[k])), \quad k = N, \ldots, 1, \\
\mathcal{P}^-[k] = \begin{cases} \mathcal{P}^{1-}[k] \ \text{ if } \ \mathcal{P}^{1-}[k] \subseteq \mathcal{Y}[k], \\ \boldsymbol{P}^-_{p^-[k],P^-[k]}(\mathcal{P}^{1-}[k] \bigcap \mathcal{Y}[k]) \ \text{ otherwise,} \end{cases} \quad k = N-1, \ldots, 0, \quad \mathcal{P}^-[N] = \mathcal{M}.
\end{gathered} \tag{3.2}$$

Here $\Gamma[\cdot]$, $P^-[\cdot]$, $p^-[\cdot]$ are parameters that are assumed to be admissible, i. e.,

$$\Gamma[k] \in \mathcal{G}^{n_u \times n}, \ k = N, \ldots, 1, \quad \det P^-[k] \neq 0, \quad p^-[k] \in \operatorname{int}(\mathcal{P}^{1-}[k] \textstyle\bigcap \mathcal{Y}[k]), \ k = N-1, \ldots, 0.$$

Let this system be solved for some fixed admissible parameters. Consider following control strategies, where relative coordinates with respect to the cross-sections of $\mathcal{P}^{1-}[\cdot]$ are used:

$$u^j[k,x] = r[k] + \bar{R}[k]\lambda^j[k,x]\Gamma[k]\zeta^{1-}[k-1,x], \quad \zeta^{1-}[k-1,x] = \bar{P}^{1-}[k-1]^{-1}(x - p^{1-}[k-1]) \tag{3.3}$$

with three variants of the formulas for calculation of $\lambda^j[k,x]$, $j = 1, 2, 3$:

$$\lambda^1[k,x] = 1/\max\{1, \|\Gamma[k]\zeta^{1-}[k-1,x]\|_\infty\} \in \mathbb{R}^1, \tag{3.4}$$
$$\lambda^2[k,x] = \operatorname{diag}\{1/\max\{1, |(\Gamma[k]\zeta^{1-}[k-1,x])_i|\}\} \in \mathbb{R}^{n_u \times n_u}, \tag{3.5}$$
$$\lambda^3[k,x] = 1/\max\{1, \|\zeta^{1-}[k-1,x]\|_\infty\} \in \mathbb{R}^1, \tag{3.6}$$

and also

$$\begin{gathered}
u^4[k,x] = r[k] + \bar{R}[k]\Gamma[k]\lambda^4[k,x]\zeta^{1-}[k-1,x], \\
\zeta^{1-}[k-1,x] = \bar{P}^{1-}[k-1]^{-1}(x - p^{1-}[k-1]), \\
\lambda^4[k,x] = \operatorname{diag}\{1/\max\{1, |\zeta^{1-}_i[k-1,x]|\}\} \in \mathbb{R}^{n \times n}.
\end{gathered} \tag{3.7}$$

Theorem 3.2. *Let system (1.1)–(1.3) be considered under Assumptions 1.2, 1.3. Let $\Gamma[\cdot]$, $P^-[\cdot]$, $p^-[\cdot]$ be arbitrary admissible parameters and, when solving system (3.2), the following relations be satisfied:*

$$\mathrm{e} - \gamma^-[k] \geq 0, \quad \gamma^-[k] := (\operatorname{Abs}(\bar{P}^-[k]^{-1}C[k]\bar{Q}[k]))\mathrm{e}, \quad k = N, \ldots, 1, \tag{3.8}$$
$$\det \bar{P}^{1-}[k] \neq 0, \quad k = N-1, \ldots, 0, \tag{3.9}$$
$$\operatorname{int}(\mathcal{P}^{1-}[k] \textstyle\bigcap \mathcal{Y}[k]) \neq \varnothing, \quad k = N-1, \ldots, 0. \tag{3.10}$$

Then system (3.2) determines the tubes $\mathcal{P}^-[\cdot] = \mathcal{P}[p^-[\cdot], \bar{P}^-[\cdot]]$ and $\mathcal{P}^{1-}[\cdot] = \mathcal{P}[p^{1-}[\cdot], \bar{P}^{1-}[\cdot]]$ with nondegenerate cross-sections, $\mathcal{P}^-[\cdot]$ is an internal estimate for $\mathcal{W}[\cdot]$ (i. e., $\mathcal{P}^-[k] \subseteq \mathcal{W}[k]$, $k = 0, \ldots, N$), and $\mathcal{P}^-[\cdot]$ together with each of four control strategies (3.3)–(3.7) gives a particular solution to Problem 3; for every initial point $x[0] = x^0 \in \mathcal{P}^-[0]$ all these control strategies $u^j[\cdot,\cdot]$ turn out to coincide with $u[\cdot,\cdot]$ determined by the formulas

$$u[k,x] = r[k] + \bar{R}[k]\Gamma[k]\zeta^{1-}[k-1,x], \tag{3.11}$$

where $\zeta^{1-}[k-1,x]$ is defined in (3.3), and we have $x[k] \in \mathcal{P}^-[k]$, $k = 1, \ldots, N$.

P r o o f. For u^1 with (3.4), the theorem is a special case of [16, Theorem 3.1], where the formulas for tubes given in [16] follow from the formulas for $\mathcal{P}^1 \dot{-} \mathcal{P}^2$ and $\boldsymbol{P}^-_{I,\Gamma}(\mathcal{P}^1 + \mathcal{P}^2)$. The proof for u^2, u^3, and u^4 is similar to [16]. All these u^j satisfy $u^j[k,x] \in \mathcal{R}[k]$ for every $x \in \mathbb{R}^n$, and it suffices to take into account that if $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^{1-}[k-1]$, then $|\zeta^{1-}_i[k-1,x]| \leq \|\zeta^{1-}[k-1,x]\|_\infty \leq 1$ and we come to (3.11). □

Let us investigate properties of the above tubes in more detail. Introduce a notation:

$$\alpha^-[k] = (\mathrm{Abs}\,(\bar P^-[k]^{-1}B[k]\bar R[k]))\,\mathrm{e}, \quad \gamma^-[k] = (\mathrm{Abs}\,(\bar P^-[k]^{-1}C[k]\bar Q[k]))\,\mathrm{e}.$$

Proposition 3.1. *Specific formulas for the matrices of the parallelotopes in* (3.2) *are*

$$\bar P^{0-}[k-1] = \bar P^-[k]\,\mathrm{diag}\,(\mathrm{e}-\gamma^-[k]), \quad \bar P^{1-}[k-1] = A[k]^{-1}(\bar P^{0-}[k-1]-B[k]\bar R[k]\Gamma[k]). \quad (3.12)$$

Theorem 3.2 is also true under conditions (3.13)*,* (3.10)*, i. e., if relations* (3.8)*,* (3.9) *are replaced by* (3.13)*:*

$$\mathrm{e}-\gamma^-[k]-\alpha^-[k] > 0, \quad k = N,\dots,1. \quad (3.13)$$

Here the relation $\mathrm{e}-\gamma^-[k] > 0$ *guarantees nondegeracy of* $\mathcal{P}^{0-}[k-1]$ *and* $\mathrm{e}-\gamma^-[k]-\alpha^-[k] > 0$ *gives a sufficient condition for* $\mathcal{P}^{1-}[k-1]$ *to be a nondegenerate parallelotope for any* $\Gamma[k] \in \mathcal{G}^{n_u\times n}$.

P r o o f. Relations (3.12) follow from the formulas for $\mathcal{P}^1 \dot{-} \mathcal{P}^2$ and $\boldsymbol{P}^-_{I,\Gamma}(\mathcal{P}^1+\mathcal{P}^2)$. Note that (3.13) implies that $\mathrm{e}-\gamma^-[k] > 0$. Then, using (3.12) and Proposition 2.1, the sufficient condition for nondegeneracy of $\boldsymbol{P}^-_{I,\Gamma[k]}(\mathcal{P}^{0-}[k-1]+(-B[k]\mathcal{R}[k]))$ can be presented in the form

$$\mathrm{e} > (\mathrm{Abs}\,((\mathrm{diag}\,(\mathrm{e}-\gamma^-[k]))^{-1}\bar P^-[k]^{-1}B[k]\bar R[k]))\,\mathrm{e} = (\mathrm{diag}\,(\mathrm{e}-\gamma^-[k]))^{-1}\alpha^-[k].$$

The obtained inequality turns out to be equivalent to (3.13) and entails (3.8), (3.9). □

Remark 3.1. Note, similarly to [16, Remark 3.3], that conditions (3.13) can be especially useful for systems obtained by Euler's approximations of differential systems.

Theorem 3.3. *Let system* (1.1)*,* (1.2) *without state constraints* (1.3) *be considered under Assumptions 1.2, 1.3. Let* $\Gamma[\cdot]$ *be arbitrary admissible parameter, and when solving system* (3.2) *with* $\mathcal{Y}[k] \equiv \mathbb{R}^n$*, relations* (3.8)–(3.9) *(or* (3.13)*) be satisfied, and, therefore, the tube* $\mathcal{P}^-[\cdot] = \mathcal{P}[p^-[\cdot],\bar P^-[\cdot]]$ *with* $\det \bar P^-[k] \neq 0$ *be constructed. If* $x[\cdot]$ *is the trajectory that corresponds to an arbitrary initial point* $x[0] = x^0 \in \mathcal{P}^-[0]$*, to any of four strategies* $u^j[\cdot,\cdot]$ *from* (3.3)–(3.7)*, where* $\zeta^{1-}[k-1,x]$ *should be replaced by* $\zeta^-[k-1,x] = \bar P^-[k-1]^{-1}(x-p^-[k-1])$*, and to arbitrary admissible control* $v[\cdot]$*, then we have* $x[k] \in \mathcal{P}^-[k]$*,* $k = 1,\dots,N$*, and, moreover, the following guaranteed estimates in terms of relative coordinates* $\zeta^-[k] = \bar P^-[k]^{-1}(x[k]-p^-[k])$ *are valid that ensure that* $x[\cdot]$ *is inside the tube* $\mathcal{P}^-[\cdot]$ *and* $x[N] \in \mathcal{M}$*:*

$$\mathrm{Abs}\,\zeta^-[k] \le \mathrm{e} - \prod_{l=1}^{k} \mathrm{diag}\,(\mathrm{e}-\gamma^-[l]) \cdot (\mathrm{e}-\mathrm{Abs}\,\zeta^-[0]), \quad k = 1,\dots,N. \quad (3.14)$$

P r o o f. We have $\mathrm{e}-\mathrm{Abs}\,\zeta^-[0] \ge 0$ due to $x[0] \in \mathcal{P}^-[0]$ and Lemma 2.2. Following the scheme of the proof of [15, Theorem 3] we can obtain the next inequalities for components of $\zeta^-[k]$ at each time step $k \in \{1,\dots,N\}$:

$$|\zeta_i^-[k]| \le |1-\gamma_i^-[k]|\cdot|\zeta_i^-[k-1]| + \gamma_i^-[k] = 1 + (1-\gamma_i^-[k])\cdot|\zeta_i^-[k-1]| - (1-\gamma_i^-[k]) = \\ = 1-(1-\gamma_i^-[k])\cdot(1-|\zeta_i^-[k-1]|),$$

which lead to $\mathrm{e}-\mathrm{Abs}\,\zeta^-[k] \ge \prod_{l=1}^{k}\mathrm{diag}\,(\mathrm{e}-\gamma^-[l])\cdot(\mathrm{e}-\mathrm{Abs}\,\zeta^-[0]) \ge 0.$ □

Let us describe external estimates for $\mathcal{W}[\cdot]$. Consider the polyhedral tubes that satisfy

$$\begin{aligned}\mathcal{P}^{0+}[k-1] &= \mathcal{P}^{+}[k]\dot{-}C[k]\mathcal{Q}[k], \quad k=N,\dots,1,\\ \mathcal{P}^{1+}[k-1] &= A[k]^{-1}\boldsymbol{P}^{+}_{\tilde{P}^{1+}[k-1]}(\mathcal{P}^{0+}[k-1]+(-B[k]\mathcal{R}[k])), \quad k=N,\dots,1,\\ \mathcal{P}^{+}[k-1] &= \tilde{\boldsymbol{P}}^{+}_{P^{+}[k-1]}(\mathcal{P}^{1+}[k-1]\bigcap\mathcal{Y}[k-1]), \quad k=N,\dots,1, \quad \mathcal{P}^{+}[N]=\boldsymbol{P}^{+}_{P^{+}[N]}(\mathcal{M}),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\mathcal{P}^{+}[\cdot] &= \mathcal{P}(p^{+}[\cdot],P^{+}[\cdot],\pi^{+}[\cdot]) = \mathcal{P}[p^{+}[\cdot],\bar{P}^{+}[\cdot]],\\ \mathcal{P}^{1+}[\cdot] &= \mathcal{P}(p^{1+}[\cdot],P^{1+}[\cdot],\pi^{1+}[\cdot]) = \mathcal{P}[p^{1+}[\cdot],\bar{P}^{1+}[\cdot]],\\ \mathcal{P}^{0+}[\cdot] &= \mathcal{P}(p^{0+}[\cdot],P^{0+}[\cdot],\pi^{0+}[\cdot]) = \mathcal{P}[p^{0+}[\cdot],\bar{P}^{0+}[\cdot]].\end{aligned}$$

Note that in the second line of (3.15), the value of the orientation matrix of $\mathcal{P}^{1+}[k-1]$ is equal to $A[k]^{-1}\tilde{P}^{1+}[k-1]$;

$$\mathcal{P}^{+}[N] = \mathcal{P}(p_{\mathrm{f}},P^{+}[N],(\mathrm{Abs}\,(P^{+}[N]^{-1}P_{\mathrm{f}}))\,\pi_{\mathrm{f}}) = \mathcal{P}[p_{\mathrm{f}},P^{+}[N]\mathrm{diag}\,((\mathrm{Abs}\,(P^{+}[N]^{-1}\bar{P}_{\mathrm{f}}))\,\mathrm{e})].$$

So, (3.15) describe a family of polyhedral tubes generated by the admissible matrix functions $P^{+}[\cdot]$, $\tilde{P}^{1+}[\cdot]$. The following statements follow from the comparison of (3.15) with (3.1) and from the properties of the elementary estimates from Sec. 2.

Proposition 3.2. *Let system* (1.1)–(1.3) *be considered under Assumption* 1.2*, the tube* $\mathcal{W}[\cdot]$ *be determined from* (3.1)*, and* $\mathcal{P}^{+}[\cdot]$ *be determined from* (3.15) *for arbitrary nonsingular matrices* $\tilde{P}^{1+}[k]$*,* $k=N-1,\dots,0$*, and admissible matrices* $P^{+}[k]$*,* $k=N,\dots,0$*. Then* $\mathcal{P}^{+}[\cdot]$ *is external estimate for* $\mathcal{W}[\cdot]$ *(i. e.,* $\mathcal{W}[k]\subseteq\mathcal{P}^{+}[k]$*,* $k=N,\dots,0$*), and if* $\mathcal{P}^{+}[k]=\varnothing$ *for some* $k\in\{N-1,\dots,0\}$*, then Problem* 1 *is unsolvable on the time interval* $[0,N]$*.*

§ 4. Solving evasion problems 2 and 4

To solve Problem 2 let us consider the following system of recurrence relations for construction of the tubes $\hat{\mathcal{W}}[\cdot]$, $\hat{\mathcal{W}}^{0}[\cdot]$, and $\hat{\mathcal{W}}^{1}[\cdot]$:

$$\begin{aligned}\hat{\mathcal{W}}^{0}[k-1] &= \hat{\mathcal{W}}[k]+(-B[k]\mathcal{R}[k]), \quad k=N,\dots,1,\\ \hat{\mathcal{W}}^{1}[k-1] &= A[k]^{-1}(\hat{\mathcal{W}}^{0}[k-1]\dot{-}C[k]\mathcal{Q}[k]), \quad k=N,\dots,1,\\ \hat{\mathcal{W}}[k] &= \hat{\mathcal{W}}^{1}[k]\bigcap\mathcal{Y}[k], \quad k=N-1,\dots,0, \quad \hat{\mathcal{W}}[N]=\mathcal{M},\end{aligned} \tag{4.1}$$

and the corresponding control strategies:

$$v[k,x]\in\begin{cases}\mathcal{V}[k,x]=\mathcal{Q}[x]\bigcap\{v\,|\,C[k]v\in(\mathbb{R}^{n}\backslash\hat{\mathcal{W}}^{0}[k-1])-A[k]x\} & \text{for } x\notin\hat{\mathcal{W}}^{1}[k-1],\\ \mathcal{Q}[k], & \text{otherwise.}\end{cases} \tag{4.2}$$

Theorem 4.1. *Let* $\hat{\mathcal{W}}[\cdot]$*,* $\hat{\mathcal{W}}^{0}[\cdot]$*, and* $\hat{\mathcal{W}}^{1}[\cdot]$ *satisfy* (4.1)*. Then* $\hat{\mathcal{W}}[\cdot]$ *together with any control strategy* $v[k,x]$ *that satisfies* (4.2) *gives a solution to Problem* 2*, and the tube* $\hat{\mathcal{W}}[\cdot]$ *is minimal.*

P r o o f. Let us construct $\check{\mathcal{W}}[\cdot]$ starting from $k=N$. Suppose $\check{\mathcal{W}}[k]$ is constructed. Then $\check{\mathcal{W}}[k-1]$ cannot be greater than the set of points x for which either $x\notin\mathcal{Y}[k-1]$ or there exists $v\in\mathcal{Q}[k]$ providing $A[k]x+B[k]u+C[k]v\in\check{\mathcal{W}}[k]$ for any admissible $u\in\mathcal{R}[k]$, i. e., be greater than the set $\tilde{\mathcal{W}}=A[k]^{-1}((\check{\mathcal{W}}[k]\dot{-}B[k]\mathcal{R}[k])+(-C[k]\mathcal{Q}[k]))\bigcup(\mathbb{R}^{n}\setminus\mathcal{Y}[k-1])$. Therefore we can consider the relations

$$\begin{aligned}\check{\mathcal{W}}^{0}[k-1] &= \check{\mathcal{W}}[k]\dot{-}B[k]\mathcal{R}[k],\\ \check{\mathcal{W}}^{1}[k-1] &= A[k]^{-1}(\check{\mathcal{W}}^{0}[k-1]+(-C[k]\mathcal{Q}[k])), \quad k=N,\dots,1,\\ \check{\mathcal{W}}[k-1] &= \check{\mathcal{W}}^{1}[k-1]\bigcup(\mathbb{R}^{n}\setminus\mathcal{Y}[k-1]), \quad k=N,\dots,1, \quad \check{\mathcal{W}}[N]=\mathbb{R}^{n}\setminus\mathcal{M}.\end{aligned} \tag{4.3}$$

Let a solution to (4.3) be found. Consider a trajectory $x[\cdot]$ with $x[0] \in \check{\mathcal{W}}[0]$. For some $k \in \{1,\dots,N\}$, let $x[k-1] \in \check{\mathcal{W}}[k-1]$. Then if $x[k-1] \notin \mathcal{Y}[k-1]$, then a condition from (1.7) is satisfied for $k-1$, the aim of v is achieved, and we don't need to construct v later or we can apply any $v[k] \in \mathcal{Q}[k]$. If $x[k-1] \in \mathcal{Y}[k-1]$, then, according to (4.3), $x[k-1] \in \check{\mathcal{W}}^1[k-1]$, and we can introduce a set $\check{\mathcal{V}}[k,x] = \mathcal{Q}[x] \bigcap \{v \mid C[k]v \in \check{\mathcal{W}}^0[k-1] - A[k]x\}$. Reasoning like [21, proof of Theorem 1] we can ascertain that $\check{\mathcal{V}}[k,x] \neq \varnothing$ for any $x \in \check{\mathcal{W}}^1[k-1]$, and for any $v[k,x] \in \check{\mathcal{V}}[k,x]$ and arbitrary $u[k] \in \mathcal{R}[k]$ one has $A[k]x + B[k]u[k] + C[k]v[k,x] \in \check{\mathcal{W}}[k]$. Thus, starting from $k=0$ and using such control strategies, one either comes to at least to one of the aims (1.7) or, having (1.3), comes to $x[N] \in \check{\mathcal{W}}[N]$, i. e., $x[N] \notin \mathcal{M}$.

It remains to prove that the sets $\hat{\mathcal{W}}^0[k] = \mathbb{R}^n \setminus \check{\mathcal{W}}^0[k]$, $\hat{\mathcal{W}}^1[k] = \mathbb{R}^n \setminus \check{\mathcal{W}}^1[k]$, $\hat{\mathcal{W}}[k] = \mathbb{R}^n \setminus \check{\mathcal{W}}[k]$ satisfy (4.1) and $\mathcal{V}[k,x] = \check{\mathcal{V}}[k,x]$. The formulas for $\hat{\mathcal{W}}^0[k]$, $\hat{\mathcal{W}}^1[k]$, $\mathcal{V}[k,x]$ follow from [21, proof of Proposition 2]. The formula for $\hat{\mathcal{W}}[k]$ follows from the relations of the type

$$\hat{\mathcal{W}} = \mathbb{R}^n \setminus \check{\mathcal{W}} = \mathbb{R}^n \setminus (\check{\mathcal{W}}^1 \bigcup (\mathbb{R}^n \setminus \mathcal{Y})) = (\mathbb{R}^n \setminus \check{\mathcal{W}}^1) \bigcap (\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \mathcal{Y})) = \hat{\mathcal{W}}^1 \bigcap \mathcal{Y}. \qquad \square$$

Remark 4.1. Empty cross-sections of the tubes are allowed in (4.1). If Problem 1 is solvable (we have $\mathcal{W}[k] \neq \varnothing$, $k = N,\dots,0$), then all $\hat{\mathcal{W}}[k]$ are nonempty too and $\mathcal{W}[k] \subseteq \hat{\mathcal{W}}[k]$, $k = N,\dots,0$, due to the known relation $(\mathcal{X}^1 \dot{-} \mathcal{X}^2) + \mathcal{X}^3 \subseteq (\mathcal{X}^1 + \mathcal{X}^3) \dot{-} \mathcal{X}^2$ [22, (3.1.13)].

Also note that the second line in (4.2) completes the definition of $v[k,x]$ for all x (see arguments from [21, Remark 4]), but for $x[0] \in \hat{\mathcal{W}}[0]$, generally speaking, this can not guarantee fulfilment of all the desired conditions for $x[\cdot]$.

To solve Problem 4, let us consider a family of polyhedral tubes $\hat{\mathcal{P}}^+[\cdot]$ that satisfy

$$\begin{aligned} \hat{\mathcal{P}}^{0+}[k-1] &= \boldsymbol{P}^+_{\hat{P}^{0+}[k-1]}(\hat{\mathcal{P}}^+[k] + (-B[k]\mathcal{R}[k])), \quad k = N,\dots,1, \\ \hat{\mathcal{P}}^{1+}[k-1] &= A[k]^{-1}(\hat{\mathcal{P}}^{0+}[k-1] \dot{-} C[k]\mathcal{Q}[k]), \quad k = N,\dots,1, \\ \hat{\mathcal{P}}^{+}[k-1] &= \tilde{\boldsymbol{P}}^+_{\hat{P}^{+}[k-1]}(\hat{\mathcal{P}}^{1+}[k-1] \bigcap \mathcal{Y}[k-1]), \quad k = N,\dots,1, \quad \hat{\mathcal{P}}^+[N] = \boldsymbol{P}^+_{\hat{P}^+[N]}(\mathcal{M}), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \hat{\mathcal{P}}^+[k] &= \mathcal{P}(\hat{p}^+[k], \hat{P}^+[k], \hat{\pi}^+[k]) = \mathcal{P}[\hat{p}^+[k], \hat{\hat{P}}^+[k]], \\ \hat{\mathcal{P}}^{1+}[k] &= \mathcal{P}(\hat{p}^{1+}[k], \hat{P}^{1+}[k], \hat{\pi}^{1+}[k]) = \mathcal{P}[\hat{p}^{1+}[k], \hat{\hat{P}}^{1+}[k]], \\ \hat{\mathcal{P}}^{0+}[k] &= \mathcal{P}(\hat{p}^{0+}[k], \hat{P}^{0+}[k], \hat{\pi}^{0+}[k]) = \mathcal{P}[\hat{p}^{0+}[k], \hat{\hat{P}}^{0+}[k]] \end{aligned}$$

for nonempty cross-sections. Here $\hat{\mathcal{P}}^{1+}[k]$, $\hat{\mathcal{P}}^+[k]$, and $\hat{\mathcal{P}}^{0+}[k]$ may degenerate into empty sets. In particular, if at some time step $\hat{\pi}^{1+}[k-1] < 0$ is obtained, then we accept that $\hat{\mathcal{P}}^{1+}[k-1] = \varnothing$.

Comparing (4.1) and (4.4) and using inclusion monotonicity we have the following.

Proposition 4.1. *Under Assumption 1.2, let $\hat{\mathcal{P}}^+[\cdot]$ be determined from (4.4) for arbitrary admissible matrix functions $\hat{P}^{0+}[\cdot]$ and $\hat{P}^+[\cdot]$. Then $\hat{\mathcal{P}}^+[k]$ and $\mathbb{R}^n \setminus \hat{\mathcal{P}}^+[k]$ are external and internal estimates for $\hat{\mathcal{W}}[k]$ and $\check{\mathcal{W}}[k]$ respectively for all $k = N,\dots,0$.*

Remembering Remark 4.1 it is interesting to compare the polyhedral tubes $\hat{\mathcal{P}}^+[\cdot]$ and $\mathcal{P}^+[\cdot]$ for a specific choice of the orientation matrices. Using Lemma 2.1 we obtain the following.

Proposition 4.2. *Under Assumption 1.2, let the tubes $\hat{\mathcal{P}}^+[\cdot]$ and $\mathcal{P}^+[\cdot]$ be determined by (4.4) and (3.15) respectively, where the orientation matrices involved satisfy $\hat{P}^+[N] = P^+[N]$, $\hat{P}^{0+}[k-1] = \tilde{P}^{1+}[k-1]$, $k = N,\dots,1$, $\hat{P}^+[k] = P^+[k]$, $k = N-1,\dots,0$, and the formulas of the type $\tilde{\boldsymbol{P}}^+_V(\mathcal{P} \bigcap \mathcal{Y})$ are identical. Then $\mathcal{P}^+[k] \subseteq \hat{\mathcal{P}}^+[k]$, $k = N,\dots,0$. If, in addition,*

$\hat{P}^{0+}[k-1] = \hat{P}^{+}[k]$, $\tilde{P}^{1+}[k-1] = P^{+}[k]$, $k = N, \ldots, 1$, *then the tubes* $\hat{\mathcal{P}}^{+}[\cdot]$ *and* $\mathcal{P}^{+}[\cdot]$ *coincide:* $\hat{\mathcal{P}}^{+}[k] = \mathcal{P}^{+}[k]$, $k = N, \ldots, 0$, *and the following formulas for* $\hat{\mathcal{P}}^{1+}[k-1] \neq \varnothing$ *in* (4.4) *hold*

$$\hat{P}^{1+}[k-1] = A[k]^{-1}\hat{P}^{+}[k],$$
$$\hat{\pi}^{1+}[k-1] = \hat{\pi}^{+}[k] + (\mathrm{Abs}\,(\hat{P}^{+}[k]^{-1}B[k]\bar{R}[k]))\,\mathrm{e} - (\mathrm{Abs}\,(\hat{P}^{+}[k]^{-1}C[k]\bar{Q}[k]))\,\mathrm{e}.$$

Thus the choice $\hat{P}^{0+}[k-1] = \hat{P}^{+}[k]$, $k = N, \ldots, 1$, may have some advantages.

Let us present a particular solution to Problem 4. To simplify formulas let us consider the tubes $\hat{\mathcal{P}}^{+}[\cdot]$ using the mentioned choice $\hat{P}^{0+}[k-1] = \hat{P}^{+}[k]$, $k = N, \ldots, 1$:

$$\begin{aligned} \hat{\mathcal{P}}^{0+}[k-1] &= \boldsymbol{P}^{+}_{\hat{P}^{+}[k]}(\hat{\mathcal{P}}^{+}[k] + (-B[k]\mathcal{R}[k])), \quad k = N, \ldots, 1,\\ \hat{\mathcal{P}}^{1+}[k-1] &= A[k]^{-1}(\hat{\mathcal{P}}^{0+}[k-1] \dot{-} C[k]\mathcal{Q}[k]), \quad k = N, \ldots, 1, \qquad (4.5)\\ \hat{\mathcal{P}}^{+}[k-1] &= \tilde{\boldsymbol{P}}^{+}_{\hat{P}^{+}[k-1]}(\hat{\mathcal{P}}^{1+}[k-1] \textstyle\bigcap \mathcal{Y}[k-1]), \quad k = N, \ldots, 1, \quad \hat{\mathcal{P}}^{+}[N] = \boldsymbol{P}^{+}_{\hat{P}^{+}[N]}(\mathcal{M}). \end{aligned}$$

For arbitrary $x \in \mathbb{R}^n$, let us consider the following control strategy:

$$v[k,x] = \begin{cases} q[k] + \bar{Q}[k]\chi[k,x], & \text{if } \hat{\mathcal{P}}^{+}[k] \neq \varnothing \text{ and } \hat{\mathcal{P}}^{1+}[k] \neq \varnothing,\\ q[k] + \bar{Q}[k]\chi^{\varnothing}[k,x], & \text{if } \hat{\mathcal{P}}^{+}[k] \neq \varnothing \text{ and } \hat{\mathcal{P}}^{1+}[k] = \varnothing,\\ q[k], & \text{if } \hat{\mathcal{P}}^{+}[k] = \varnothing, \end{cases}$$

$$\hat{\Theta}^{+}[k] = \hat{P}^{+}[k]^{-1}C[k]\bar{Q}[k], \quad \hat{\xi}^{1+}[k-1,x] = \hat{P}^{1+}[k-1]^{-1}(x - \hat{p}^{1+}[k-1]),$$
$$\chi_j[k,x] = \mathrm{sign}\,(\hat{\Theta}^{+}[k])^{j}_{i_*} \cdot \mathrm{sign}\,\hat{\xi}^{1+}_{i_*}[k-1,x], \quad j = 1, \ldots, n_v, \qquad (4.6)$$
$$i_* = i_*[k,x] \in \mathrm{Argmax}\,\{\mathrm{Abs}\,\hat{\xi}^{1+}_{i}[k-1,x] - \hat{\pi}^{1+}_{i}[k-1] \mid i \in \{1, \ldots, n\}\},$$
$$\chi^{\varnothing}_j[k,x] = \begin{cases} \mathrm{sign}\,(\hat{\Theta}^{+}[k])^{j}_{i_*} \cdot \mathrm{sign}\,\hat{\xi}^{1+}_{i_*}[k-1,x], & \text{if } \hat{\xi}^{1+}_{i_*}[k-1,x] \neq 0,\\ \mathrm{sign}\,(\hat{\Theta}^{+}[k])^{j}_{i_*}, & \text{if } \hat{\xi}^{1+}_{i_*}[k-1,x] = 0, \end{cases} \quad j = 1, \ldots, n_v,$$
$$i_* = i_*[k,x] \in \mathrm{Argmax}\,\{\mathrm{Abs}\,\hat{\xi}^{1+}_{i}[k-1,x] - \hat{\pi}^{1+}_{i}[k-1] \mid i \in \{1, \ldots, n\},\ \hat{\pi}^{1+}_{i}[k-1] < 0\}.$$

Theorem 4.2. *Under Assumption 1.2, let* $\hat{P}^{+}[k]$, $k = N, \ldots, 0$, *be arbitrary admissible orientation matrices and the tube* $\hat{\mathcal{P}}^{+}[\cdot]$ *satisfy* (4.5). *Then* $\hat{\mathcal{P}}^{+}[\cdot]$ *together with any control strategy* $v[\cdot,\cdot]$ *of the form* (4.6) *gives a particular solution to Problem 4.*

P r o o f. Let $\hat{\mathcal{P}}^{+}[\cdot]$ and $\hat{\mathcal{P}}^{1+}[\cdot]$ be found and $x[0] = x^0 \notin \hat{\mathcal{P}}^{+}[0]$. If $x[0] \notin \hat{\mathcal{P}}^{1+}[0]$, then basing on [21, Corollary 1] about solutions to one-step evasion problem we can see that using of the control $v[1, x[0]]$ ensures $x[1] \notin \hat{\mathcal{P}}^{+}[1]$ under any $u[1] \in \mathcal{R}[1]$. If $x[0] \notin \hat{\mathcal{P}}^{+}[0]$ but $x[0] \in \hat{\mathcal{P}}^{1+}[0]$, then we have $x[0] \notin \mathcal{Y}[0]$ because otherwise we get $x[0] \in \hat{\mathcal{P}}^{1+}[0]\mathcal{Y}[0] \subseteq \hat{\mathcal{P}}^{+}[0]$, a contradiction. Therefore we already got (1.7) for $k = 0$. Repeating the arguments successively for all $k = 1, \ldots, N$ we conclude that for trajectories $x[\cdot]$ with $x[0] \notin \hat{\mathcal{P}}^{+}[0]$ we obtain that either $x[\cdot]$ is outside $\hat{\mathcal{P}}^{+}[\cdot]$ and $x[N] \notin \mathcal{M}$, or (1.7) holds for some $k \in \{1, \ldots, N\}$. □

Remark 4.2. In terms of parallelotopes, the specific formulas for the matrices of the parallelotopes $\hat{\mathcal{P}}^{1+}[k-1]$ in (4.5) can be presented in the following form:

$$\hat{\bar{P}}^{1+}[k-1] = A[k]^{-1}\,\hat{\bar{P}}^{+}[k]\,(I + \mathrm{diag}\,\hat{\alpha}^{+}[k] - \mathrm{diag}\,\hat{\gamma}^{+}[k]),$$
$$\hat{\alpha}^{+}[k] = (\mathrm{Abs}\,(\hat{\bar{P}}^{+}[k]^{-1}B[k]\bar{R}[k]))\,\mathrm{e}, \quad \hat{\gamma}^{+}[k] = (\mathrm{Abs}\,(\hat{\bar{P}}^{+}[k]^{-1}C[k]\bar{Q}[k]))\,\mathrm{e}.$$

Remark 4.3. Using [21, Corollary 1] one can also construct the solutions to Problem 4 basing on the tubes from the wider family of the polyhedral tubes determined by relations (4.4). Also, for the tubes with nondegenerate cross-sections one can apply the control strategies that are similar to $v^I[k,x]$ from [17, Theorem 1] and $v^2[k,x]$ from [21, Theorem 2], where relative coordinates of the point x calculated relative to $\hat{\mathcal{P}}^{1+}[k-1]$ should be used.

Remark 4.4. Relations (4.5) define the family of the tubes $\hat{\mathcal{P}}^+[\cdot]$ with the parameter $\hat{P}^+[\cdot]$. If concretize a rule to choose $\hat{P}^+[k]$ for $k<N$, we can obtain a subfamily of the tubes determined only by $P_{\mathrm{f}}^+ := \hat{P}^+[N]$. Thus we can introduce 4 subfamilies $\hat{\mathfrak{P}}^{+,(l)}$, $l = 1, \ldots, 4$, corresponding to the mentioned in Sec. 2 ways to construct estimates $\tilde{\boldsymbol{P}}_V^+(\mathcal{P} \bigcap \mathcal{S})$ and to choose V. Namely, $\hat{\mathfrak{P}}^{+,(1)}$ corresponds to (2.1) with Choice I, $\hat{\mathfrak{P}}^{+,(2)}$ to (2.1) with Choice III, $\hat{\mathfrak{P}}^{+,(3)}$ and $\hat{\mathfrak{P}}^{+,(4)}$ correspond to $\tilde{\boldsymbol{P}}_V^+(\mathcal{P}^1 \bigcap \mathcal{P}^2) = \boldsymbol{P}_V^+(\mathcal{P}^1) \bigcap \boldsymbol{P}_V^+(\mathcal{P}^2)$ with Choice I and Choice II respectively.

There is some analog of Theorem 3.3, namely, we have the following.

Theorem 4.3 (see [17, Theorem 1]). *Let system* (1.1)*,* (1.2) *without state constraints be considered under Assumptions* 1.2 *and* 1.3*. Let $\hat{P}^+[N]$ be arbitrary nonsingular matrix and, when solving system* (4.5) *with $\mathcal{Y}[k] \equiv \mathbb{R}^n$, the relations*

$$\mathrm{e} + \hat{\alpha}^+[k] - \hat{\gamma}^+[k] > 0, \quad k = N, \ldots, 1,$$

hold and hence the tube $\hat{\mathcal{P}}^+[\cdot]$ with nondegenerate cross-sections be constructed. If $x[\cdot]$ is the trajectory that corresponds to an arbitrary initial point $x[0] = x^0 \notin \hat{\mathcal{P}}^+[0]$, to $v[\cdot,\cdot] = v^I[\cdot,\cdot]$ from [17, Theorem 1]*, and to arbitrary admissible $u[\cdot]$, then we have $x[k] \notin \hat{\mathcal{P}}^+[k]$, $k = 1, \ldots, N$, and, moreover, the following guaranteed estimates in terms of relative coordinates $\hat{\zeta}^+[k] = \hat{\bar{P}}^+[k]^{-1}(x - \hat{p}^+[k])$ are valid for $k = 1, \ldots, N$, that ensure that $x[\cdot]$ is outside $\hat{\mathcal{P}}^+[\cdot]$ and $x[N] \notin \mathcal{M}$:*

$$\|\hat{\zeta}^+[k]\|_\infty - 1 \geq (\|\hat{\zeta}^+[0]\|_\infty - 1) \prod_{l=1}^{k} \min_{1 \leq i \leq n} (1 + \hat{\alpha}_i^+[l] - \hat{\gamma}_i^+[l]), \quad k = 1, \ldots, N. \tag{4.7}$$

Note that (3.14) gives estimates for all components of $\zeta^-[k]$, while (4.7) only for $\|\hat{\zeta}^+[k]\|_\infty$.

Let us describe internal estimates $\hat{\mathcal{P}}^-[\cdot]$ for $\hat{\mathcal{W}}[\cdot]$. Consider the following relations:

$$\begin{aligned}
&\hat{\mathcal{P}}^{0-}[k-1] = \boldsymbol{P}^-_{I,\hat{\Gamma}[k]}(\hat{\mathcal{P}}^-[k] + (-B[k]\mathcal{R}[k])), \quad k = N, \ldots, 1,\\
&\hat{\mathcal{P}}^{1-}[k-1] = A[k]^{-1}(\hat{\mathcal{P}}^{0-}[k-1] \dot{-} C[k]\mathcal{Q}[k]), \quad k = N, \ldots, 1,\\
&\hat{\mathcal{P}}^-[k] = \begin{cases} \hat{\mathcal{P}}^{1-}[k] \ \text{ if } \ \hat{\mathcal{P}}^{1-}[k] \subseteq \mathcal{Y}[k], \\ \boldsymbol{P}^-_{\hat{p}^-[k],\hat{P}^-[k]}(\hat{\mathcal{P}}^{1-}[k] \bigcap \mathcal{Y}[k]) \ \text{ otherwise}, \end{cases} \quad k = N-1, \ldots, 0, \quad \hat{\mathcal{P}}[N] = \mathcal{M},
\end{aligned} \tag{4.8}$$

where the formulas for the matrices of the parallelotopes $\hat{\mathcal{P}}^{0-}[k-1]$, $\hat{\mathcal{P}}^{1-}[k-1]$ in (4.8) are:

$$\begin{aligned}
\hat{\bar{P}}^{0-}[k-1] &= \hat{\bar{P}}^-[k] - B[k]\bar{R}[k]\hat{\Gamma}[k],\\
\hat{\bar{P}}^{1-}[k-1] &= A[k]^{-1}\hat{\bar{P}}^{0-}[k-1] \cdot (\mathrm{diag}\,(\mathrm{e} - (\mathrm{Abs}\,(\hat{\bar{P}}^{0-}[k-1]^{-1}C[k]\bar{Q}[k]))\mathrm{e})).
\end{aligned} \tag{4.9}$$

Proposition 4.3. *Under Assumptions* 1.2, 1.3*, let $\hat{\Gamma}[\cdot]$, $\hat{P}^-[\cdot]$, $\hat{p}^-[\cdot]$ be arbitrary admissible parameters, i. e., all $\hat{\Gamma}[k] \in \mathcal{G}^{n_u \times n}$, $\det \hat{P}^-[k] \neq 0$, $\hat{p}^-[k] \in \mathrm{int}\,(\hat{\mathcal{P}}^{1-}[k] \bigcap \mathcal{Y}[k])$, and let the sets $\hat{\mathcal{P}}^{0-}[k]$, $\hat{\mathcal{P}}^{1-}[k]$, and $\hat{\mathcal{P}}^-[k]$ turn out to be nondegenerate parallelotopes for all $k = N-1, \ldots, 0$. Then system* (4.8) *determines the tube $\hat{\mathcal{P}}^-[\cdot]$, which is an internal estimate for $\hat{\mathcal{W}}[\cdot]$ (i. e., $\hat{\mathcal{P}}^-[k] \subseteq \hat{\mathcal{W}}[k]$, $k = 0, \ldots, N$). The simple sufficient conditions for nondegeneracy of $\hat{\mathcal{P}}^{0-}[k-1]$ and $\hat{\mathcal{P}}^{1-}[k-1]$ under any $\hat{\Gamma}[k] \in \mathcal{G}^{n_u \times n}$ and $\det \hat{\bar{P}}^-[k] \neq 0$ are as follows:*

$$\begin{gathered}
\|\hat{\alpha}^-[k]\|_\infty + \|\hat{\gamma}^-[k]\|_\infty < 1, \quad k = N, \ldots, 1,\\
\hat{\alpha}^-[k] := (\mathrm{Abs}\,(\hat{\bar{P}}^-[k]^{-1}B[k]\bar{R}[k]))\,\mathrm{e}, \quad \hat{\gamma}^-[k] := (\mathrm{Abs}\,(\hat{\bar{P}}^-[k]^{-1}C[k]\bar{Q}[k]))\,\mathrm{e}.
\end{gathered}$$

P r o o f. Inclusions $\hat{\mathcal{P}}^{-}[k] \subseteq \hat{\mathcal{W}}[k]$ and formulas (4.9) are obtained from (4.1) using arguments of inclusion monotonicity and formulas for the elementary polyhedral estimates. The above sufficient conditions are obtained using the following lemma. □

Lemma 4.1. *Let matrices P^1 and P^2 be such that*

$$P^1 = P - H^1\Gamma, \quad P^2 = \operatorname{diag}(\mathrm{e} - (\operatorname{Abs}((P^1)^{-1}H^2))\mathrm{e}),$$

where $\det P \neq 0$, $\|\Gamma\|_\infty \leq 1$. *Denote* $\alpha = (\operatorname{Abs}\Xi)\,\mathrm{e}$, $\Xi = P^{-1}H^1$, $\gamma = (\operatorname{Abs}\Theta)\mathrm{e}$, $\Theta = P^{-1}H^2$. *If* $\|\alpha\|_\infty + \|\gamma\|_\infty < 1$, *then both* $\det P^1 \neq 0$ *and* $\det P^2 \neq 0$.

P r o o f. Proof is based on the theorem from [23, Sec. 7.1] which says that if $\|\cdot\|$ denotes any matrix norm for which $\|I\| = 1$ and if $\|M\| < 1$, then $(I+M)^{-1}$ exists and $\|(I+M)^{-1}\| \leq$ $\leq 1/(1-\|M\|)$. Nonsingularity of P^1 follows from the relations $P^1 = P(I - \Xi\Gamma)$, $\|\Xi\Gamma\|_\infty \leq$ $\leq \|\Xi\|_\infty \cdot 1 = \|\alpha\|_\infty < 1$. Nonsingularity of P^2 follows from the relations

$$(P^1)^{-1}H^2 = (I - \Xi\Gamma)^{-1}\Theta,$$
$$\|(\operatorname{Abs}((P^1)^{-1}H^2))\,\mathrm{e}\|_\infty \leq \|(I - \Xi\Gamma)^{-1}\|_\infty \cdot \|\Theta\|_\infty \cdot 1 \leq 1/(1 - \|\alpha\|_\infty) \cdot \|\gamma\|_\infty < 1. \quad □$$

In the next Sect. 5, we consider an example to illustrate the following theoretically proved general situation. Suppose one has constructed several pairs of tubes $\mathcal{P}^{-,\alpha}[\cdot]$, $\mathcal{P}^{1-,\alpha}[\cdot]$ using (3.2) and several pairs of tubes $\hat{\mathcal{P}}^{+,\beta}[\cdot]$, $\hat{\mathcal{P}}^{1+,\beta}[\cdot]$ using (4.5). Given $x[0] = x^0$. If $x[0] \in \bigcup_\alpha \mathcal{P}^{-,\alpha}[0]$, then usage of u based on $\mathcal{P}^{-,\alpha_*}[\cdot]$, $\mathcal{P}^{1-,\alpha_*}[\cdot]$ such that $x^0 \in \mathcal{P}^{-,\alpha_*}[0]$ ensures $x[N] \in \mathcal{M}$ and (1.3). If $x^0 \in \bigcup_\beta (\mathbb{R}^n \setminus \hat{\mathcal{P}}^{+,\beta}[0])$, then usage of v based on $\hat{\mathcal{P}}^{+,\beta_*}[\cdot]$, $\hat{\mathcal{P}}^{1+,\beta_*}[\cdot]$ such that $x^0 \notin \hat{\mathcal{P}}^{+,\beta_*}[0]$ guarantees either $x[N] \notin \mathcal{M}$ or $x[k] \notin \mathcal{Y}[k]$ for some $k \in \{0,\ldots,N-1\}$.

Remark 4.5. Suppose one has constructed several pairs of the tubes $\hat{\mathcal{P}}^{+,\beta}[\cdot]$ and $\hat{\mathcal{P}}^{1+,\beta}[\cdot]$ using (4.5). Then the evasion aim (i. e., $x[N] \notin \mathcal{M}$ or $x[k] \notin \mathcal{Y}[k]$ for some $k \in \{0,\ldots,N-1\}$) is also achievable for each initial point x^0 from the set $(\bigcup_\beta (\mathbb{R}^n \setminus \hat{\mathcal{P}}^{1+,\beta}[0])) \bigcup (\mathbb{R}^n \setminus \mathcal{Y}[0])$, which, generally speaking, can be greater than the mentioned set $\bigcup_\beta (\mathbb{R}^n \setminus \hat{\mathcal{P}}^{+,\beta}[0])$ if $\mathcal{Y}[0] \neq \mathbb{R}^n$. Note that it is very easy to check whether x^0 belongs to the zone $\mathcal{Y}[0]$ or not. Similarly, the approach aim (i. e., $x[N] \in \mathcal{M}$ under (1.3)) is achievable for each initial point x^0 from the set $(\bigcup_\alpha \mathcal{P}^{1-,\alpha}[0]) \bigcap \mathcal{Y}[0]$, which can be greater than the mentioned set $\bigcup_\alpha \mathcal{P}^{-,\alpha}[0]$.

§ 5. Example

Let $n = 2$ (for ease of visualization). Consider the system obtained by the Euler discretization of a differential one determined on a time interval $[0, \theta]$:

$$A[k] \equiv I + h_N \cdot \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}, \quad B[k] \equiv h_N \cdot (0,1)^\top, \quad \mathcal{R}[k] \equiv \mathcal{P}(0, I, 1) \subseteq \mathbb{R}^1, \quad C[k] \equiv h_N \cdot (1,0)^\top,$$
$$\mathcal{Q}[k] \equiv \mathcal{P}(0, I, 0.2) \subseteq \mathbb{R}^1, \quad \mathcal{M} = \mathcal{P}((-0.5,0)^\top, I, (0.5,0.5)^\top), \quad h_N = \theta/N, \ \theta = 2, \ N = 200,$$

under the state constraints of the form $|x_1 + 0.2| \leq 0.8$, $|x_2| \leq 2.1$.

We have found three tubes $\mathcal{P}^{-,\alpha}[\cdot]$ as in [16, Example 5.1] and four tubes $\hat{\mathcal{P}}^{+,\beta,(l)}[\cdot]$ of each subfamily $\hat{\mathfrak{P}}^{+,(l)}$, $l = 1,\ldots,4$, from Remark 4.4 taking matrices P_{f}^{+} as in [24, Sec. 4].

In Fig. 1, the set $\mathcal{M}$ and state constraints are shown by dashed lines. The cross-sections $\mathcal{P}^{-,\alpha}[0]$ are presented by thin lines (see also [16, Fig. 2]), $\hat{\mathcal{P}}^{+,\beta,(l)}[0]$ by thick lines.

Namely, the cross-sections $\hat{\mathcal{P}}^{+,\beta,(l)}[0]$ for $l = 1,2,3,4$ are presented in Fig. 1, *a*, Fig. 1, *b*, Fig. 1, *c*, Fig. 1, *d*; their volumes are equal to 1.6486, 2.7237, 3.5673, 2.6630; 1.8938, 2.8815, 3.3774, 3.0197; 1.6509, 2.8025, 3.8901, 2.7295; 1.6509, 3.0099, 3.7256, 2.7510 respectively. The

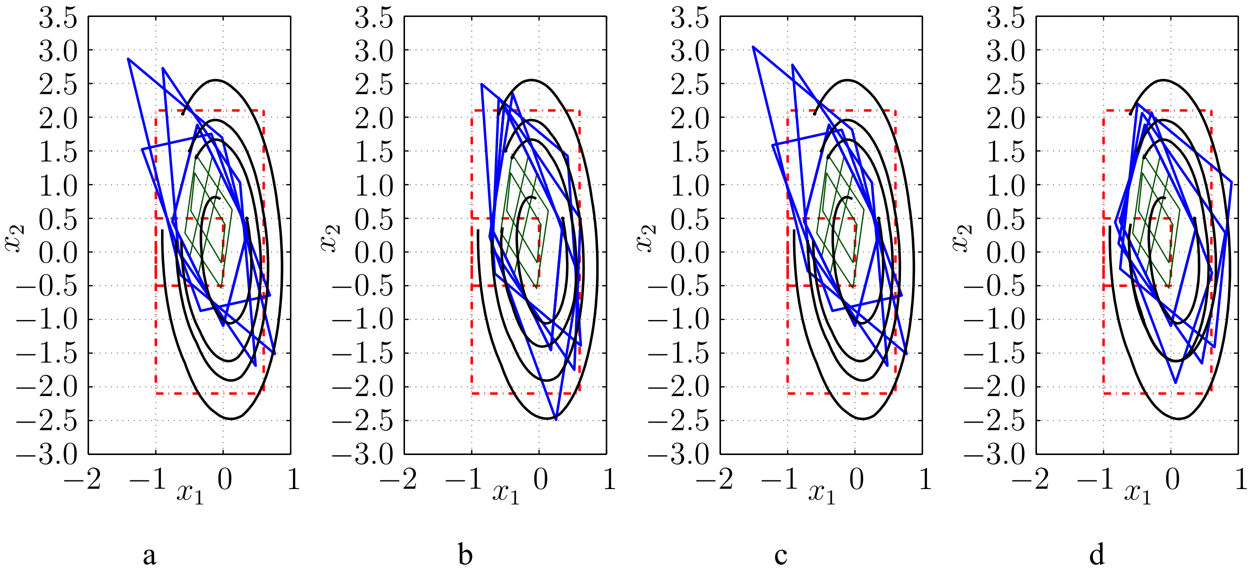

Fig. 1. Results of polyhedral synthesis for 4 initial points $x^{0,i}$ using u from (3.6) based on $\mathcal{P}^{-,\alpha}[\cdot]$ and v based on the tubes $\hat{\mathcal{P}}^{+,\beta,(l)}[\cdot]$ from $\hat{\mathfrak{P}}^{+,(1)}$ (a), $\hat{\mathfrak{P}}^{+,(2)}$ (b), $\hat{\mathfrak{P}}^{+,(3)}$ (c), $\hat{\mathfrak{P}}^{+,(4)}$ (d)

volumes of the sets $\hat{\mathcal{P}}^{+,\beta}[0]$ constructed as in [24, Sec. 4] for the case without state constraints are equal to 2.4397, 3.5118, 3.8901, 3.3877 (see also [24, Fig. 1(a)]). The minimal sets for each series over β correspond to the tubes with $P_{\rm f}^{+} = P_{\rm f}$. We see that $\hat{\mathcal{P}}^{+,\beta,(l)}[0]$ turn out to be smaller (in terms of volume) than corresponding $\hat{\mathcal{P}}^{+,\beta}[0]$. It cannot be said that some subfamily $\hat{\mathfrak{P}}^{+,(l)}$ is the best.

We consider 4 initial points $x^{0,i}$: $x^{0,1}{=}(-0.6, 2.05)^\top$, $x^{0,2}{=}(0.35, 0.5)^\top$, $x^{0,3}{=}(-0.4, 1.4)^\top$, $x^{0,4}{=}(-0.5, 1.5)^\top$ and construct corresponding trajectories $x^{i,(l)}[\cdot]$, $i{=}1,\dots,4$, $l{=}1,\dots,4$, under controls u of the form (3.3), (3.6) based on the tubes $\mathcal{P}^{-,\alpha}[\cdot]$ and controls v of the form (4.6) based on the tubes $\hat{\mathcal{P}}^{+,\beta,(l)}[\cdot]$ from $\hat{\mathfrak{P}}^{+,(l)}$; they are presented for $l = 1, \dots, 4$ in Fig. 1, *a*, Fig. 1, *b*, Fig. 1, *c*, Fig. 1, *d*. We have that $x^{0,3}$ belongs to one of $\mathcal{P}^{-,\alpha}[0]$; each of $x^{0,1}$, $x^{0,2}$, and $x^{0,4}$ is outside at least one of $\hat{\mathcal{P}}^{+,\beta,(l)}[0]$ for all $l{=}1,\dots,4$. We see that all trajectories confirm the mentioned theoretically proved general situation: $x^{3,(l)}[\cdot]$ satisfy both $x^{3,(l)}[N] \in \mathcal{M}$ and (1.3); $x^{2,(l)}[N] \notin \mathcal{M}$; $x^{1,(l)}[\cdot]$ and $x^{4,(l)}[\cdot]$ violate (1.3).

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Е. К. Костоусова

О решении задач терминального сближения и уклонения для линейных многошаговых систем при фазовых ограничениях

Ключевые слова: системы с неопределенностью, синтез управлений, задача сближения, задача уклонения, полиэдральные методы, параллелотопы, параллелепипеды.

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Работа посвящена развитию полиэдральных методов решения двух задач управления линейными многошаговыми системами с неопределенностями при фазовых ограничениях — задач терминального сближения и уклонения. Они возникают в системах с двумя управлениями, где цель одного — привести траекторию на заданное конечное множество в заданный момент времени, не нарушая фазовых ограничений, цель другого — противоположна. Предполагается, что конечное множество — параллелепипед, управления стеснены параллелотопозначными ограничениями, фазовые ограничения заданы в виде полос. Представлены методы решения обеих задач с использованием полиэдральных (параллелотопо- или параллелепипедо-значных) трубок. Методы решения задачи сближения предложены автором ранее, но здесь исследуются их дополнительные свойства. В частности, для случая без фазовых ограничений найдены гарантированные оценки для траектории, обеспечивающие ее нахождение внутри трубки. Даны удобные достаточные условия, гарантирующие получение невырожденных сечений в процессе вычислений. Для задачи уклонения сначала рассматривается общая схема решения, а затем предлагаются полиэдральные методы. Приводятся и сравниваются целые параметрические семейства внешних и внутренних полиэдральных оценок трубок разрешимости обеих задач. Приведен иллюстрирующий пример.

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