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APPLICATION OF LYAPUNOV–POINCARÉ METHOD OF SMALL PARAMETER FOR NASH AND BERGE EQUILIBRIUM DESIGNING IN ONE DIFFERENTIAL TWO-PLAYER GAME

The Poincaré small parameter method is actively used in celestial mechanics, as well as in the theory of differential equations and in its important section called optimal control. In this paper, the mentioned method is used to construct an explicit form of Nash and Berge equilibrium in a differential positional game with a small influence of one of the players on the rate of change of the state vector.

Keywords: small parameter method, differential linear-quadratic noncooperative game, Nash equilibrium, Berge equilibrium.

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Introduction

In Lyapunov's stability theory there is a section called algebraic coefficient criteria. The whole idea of such criteria is to establish the stability of an unperturbed motion without solving a system of differential equations using the signs of coefficients and/or relations among them. In this paper we apply a similar approach to equilibrium design in noncooperative linear-quadratic two-player games. More specifically, based on the sign definiteness of the quadratic forms appearing in the payoff functions of players, we will answer two questions as follows.

1) Do Berge and/or Nash equilibria exist?

2) How can we construct these equilibria?

In fact, the answers to both questions are concealed in the possibility of judging the existence of a solution for a system of two matrix ordinary differential equations of the Riccati type that is extendable on the time interval of a game. For solving this problem, we will employ dynamic programming, the small parameter method and also Poincaré's theorem on analyticity (conditions under which a solution of a differential equation is analytic with respect to a parameter).

§ 1. Preliminaries

Consider a noncooperative differential positional linear-quadratic two-player game described by

$$\Gamma_2 = \langle \{1,2\}, \Sigma, \{\mathbf{U}_i\}_{i=1,2}, \{J_i(U,t_0,x_0)\}_{i=1,2} \rangle.$$

Here $\{1,2\}$ is the set of players; the n-dimensional state vector $x \in \mathbf{R}^n$ of a controlled dynamic system Σ evolves over time t in accordance with the vector ordinary differential equation

$$\dot{x} = A(t)x + u_1 + \varepsilon u_2, \quad x(t_0) = x_0, \tag{1.1}$$

where $t \in [t_0, \vartheta]$ and a terminal time instant $\vartheta > t_0 \geqslant 0$ is fixed; the position of the game Γ_2 at a time instant t is represented by a pair $(t,x) \in [t_0,\vartheta] \times \mathbf{R}^n$, where (t_0, x_0) denotes an initial position; the elements of the system matrix $A(t)$ of dimensions $n \times n$ are assumed to be continuous on $[0,\vartheta]$, and this fact will be indicated by $A(\cdot) \in C^{n\times n}[0,\vartheta]$; $u_i \in \mathbf{R}^n$ gives the control of player i; $\varepsilon > 0$ is a small parameter, and hence Γ_2 belongs to the class of differential positional games with a small influence of player 2 on the rate of change $\dot{x}(t)$ of the state vector $x(t)$.

A strategy U_i of player i is identified with an n-dimensional vector function $u_i(t,x)$ of the form $Q_i(t)x$, where $Q_i(\cdot) \in C^{n\times n}[0,\vartheta]$, and this fact will be indicated by $U_i \div u_i(t,x) = Q_i(t)x$. The set of all such strategies is

$$\mathbf{U}_i = \left\{U_i \div Q_i(t)x \quad \forall Q_i(\cdot) \in C^{n\times n}[0,\vartheta]\right\}.$$

The strategy profile of the game Γ_2 is a pair $U = (U_1, U_2) \in \mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$. Therefore, as his strategy player i has to choose a matrix $Q_i(t)$ that is continuous on $[0,\vartheta]$ $(i = 1,2)$.

A play of the game Γ_2 is organized as follows. Based on his individual considerations (see the payoff function $J_i(U, t_0, x_0)$ defined below), each player chooses and uses his strategy $U_i^* \div u_i^* = Q_i^*(t)x$ $(i = 1,2)$. As a result, the system (1.1) takes the form

$$\dot{x} = [A(t) + Q_1^*(t) + \varepsilon Q_2^*(t)]\,x, \quad x(t_0) = x_0.$$

Such a homogeneous and linear (in variable x) system with continuous (in the variable t) coefficients has a unique continuous solution $x^*(t)$ that is extendable to $[t_0,\vartheta]$ $\forall t_0 \in [0,\vartheta)$. Using $x^*(t)$ we constructed *the realizations* $u_i^*[t] = u_i^*(t, x^*(t)) = Q_i^*(t)x^*(t)$ of the strategies $U_i^* \div Q_i^*(t)x$ $(i = 1,2)$ chosen by the players. On such a continuous triplet $\{x^*(t), u_1^*[t], u_2^*[t] \mid t_0 \leqslant t \leqslant \vartheta\}$, *the payoff function of player* i is a priori defined as a quadratic functional $(i = 1,2)$:

$$J_i(U_1^*, U_2^*, t_0, x_0) = [x^*(\vartheta)]' C_i x^*(\vartheta) + \int_{t_0}^{\vartheta} \left\{(u_1^*[t])' D_{i1} u_1^*[t] + (u_2^*[t])' D_{i2} u_2^*[t]\right\} dt. \tag{1.2}$$

The value of (1.2) is called *the payoff* of player i. In (1.2), the prime means transposition, and the matrices C_i and D_{ij} of dimensions $n\times n$ are assumed to be symmetric without loss of generality. Other notations involved include the following: 0_n is a null n-dimensional column vector; $u_i = \left(u_i^{(1)},\dots,u_i^{(n)}\right) \in \mathbf{R}^n$ $(i = 1,2)$; $V = (V_1, V_2)$; E_n and $O_{n\times n}$ are the identity and null matrices, respectively, of dimensions $n \times n$; $\det B$ is the determinant of a matrix B of dimensions $n\times n$. In addition, the gradient of a scalar function $W(t,x,u_1,u_2,V)$ with respect to u_i is given by

$$\operatorname{grad}_{u_i} W(t,x,u_1,u_2,V) = \frac{\partial W}{\partial u_i} = \begin{pmatrix} \frac{\partial W}{\partial u_i^{(1)}} \\ \vdots \\ \frac{\partial W}{\partial u_i^{(n)}} \end{pmatrix}.$$

The Hessian of $W(t,x,u_1,u_2,V)$ with respect to the components $u_i \in \mathbf{R}^n$ under fixed values of all other variables is a matrix of dimensions $n\times n$ of the form

$$\frac{\partial^2 W}{\partial u_i^2} = \begin{pmatrix} \frac{\partial^2 W}{\partial u_i^{(1)}\partial u_i^{(1)}} & \dots & \frac{\partial^2 W}{\partial u_i^{(1)}\partial u_i^{(n)}} \\ \dots & \dots & \dots \\ \frac{\partial^2 W}{\partial u_i^{(n)}\partial u_i^{(1)}} & \dots & \frac{\partial^2 W}{\partial u_i^{(n)}\partial u_i^{(n)}} \end{pmatrix}.$$

For a constant and symmetric matrix D of dimensions $n\times n$, the inequality $D > 0$ $(< 0, \leqslant 0)$ means that the quadratic form $u_i' D u_i$ is positive definite (negative definite, nonnegative definite, respectively). A direct componentwise verification shows that, for a constant vector $a \in \mathbf{R}^n$,

$$\begin{aligned} &\frac{\partial}{\partial u_i}(u_i' D u_i) = (D + D')u_i, \\ &\frac{\partial}{\partial u_i}(a' D u_i) = D'a, \\ &\frac{\partial}{\partial u_i}(a' u_i) = a, \\ &\frac{\partial^2}{\partial u_i^2}(u_i' D u_i) = D + D' = \{\text{if } D = D'\} = 2D. \end{aligned} \tag{1.3}$$

For a scalar function $W(t,x,u_i)$, the denotation $\max_{u_i} W(t,x,u_i) = Idem\{u_i \to u_i(t,x)\}$ means that

$$\max_{u_i} W(t,x,u_i) = W(t,x,u_i(t,x)) \quad \forall t \in [0,\vartheta], \quad x \in \mathbf{R}^n, \tag{1.4}$$

and the identity (1.4) holds if

$$\left.\frac{\partial W(t,x,u_i)}{\partial u_i}\right|_{u_i(t,x)} = 0_n, \qquad \left.\frac{\partial^2 W(t,x,u_i)}{\partial u_i^2}\right|_{u_i(t,x)} < 0. \tag{1.5}$$

§ 2. Explicit solution of the Riccati matrix differential equation

Proposition 2.1. *Let a matrix A of dimensions $n \times n$ and also constant and symmetric matrices C and D of dimensions $n \times n$ be such that $A(\cdot) \in C^{n\times n}[0,\vartheta]$ and*

$$C < 0, \ D < 0.$$

Then the solution $\Theta(t)$ of the Riccati matrix differential equation

$$\dot{\Theta} + \Theta A(t) + A'(t)\Theta - \Theta D^{-1}\Theta = O_{n\times n}, \quad \Theta(\vartheta) = C, \tag{2.1}$$

has the form

$$\Theta(t) = [X^{-1}(t)]' \left\{ C^{-1} + \int_t^{\vartheta} X^{-1}(\tau) D^{-1} [X^{-1}(\tau)]' \, d\tau \right\}^{-1} X^{-1}(t), \tag{2.2}$$

where $X(t)$, $0 \leqslant t \leqslant \vartheta$, satisfies the matrix system

$$\dot{X} = A(t)X, \quad X(\vartheta) = E_n. \tag{2.3}$$

P r o o f. The matrix linear homogeneous system (2.3) with continuous in t coefficients has a solution $X(\cdot) \in C^{n\times n}[0,\vartheta]$ that is extendable to $[0,\vartheta]$; moreover, $\det X(t) \neq 0 \ \forall t \in [0,\vartheta]$, because this matrix of dimensions $n \times n$ represents the fundamental system of solutions for the ordinary differential vector equation $\dot{x} = A(t)x$. Then two implications are true,

$$[\det X(t) \neq 0 \ \forall t \in [0,\vartheta]] \Rightarrow \left[\exists X^{-1}(t) \ \forall t \in [0,\vartheta]\right]$$

and

$$[X(\vartheta) = E_n] \Rightarrow [X^{-1}(\vartheta) = E_n].$$

From (2.2) it follows that, at $t = \vartheta$,

$$\Theta(\vartheta) = E_n\{C^{-1} + O_{n\times n}\}^{-1}E_n = C.$$

It's known that

$$\begin{aligned} \frac{d[X^{-1}(t)]}{dt} &= -X^{-1}(t)A(t), \quad X^{-1}(\vartheta) = E_n, \\ \frac{d[X^{-1}(t)]'}{dt} &= -A'(t)[X^{-1}(t)]', \quad [X^{-1}(\vartheta)]' = E_n. \end{aligned} \tag{2.4}$$

Denote by $\{\cdots\}$ the parenthesized expression in (2.2). In view of (2.4), (2.2) and $[X^{-1}(t)]' = [X'(t)]^{-1}$ (see [9, p. 33]), differentiating both sides of (2.2) with respect to t gives

$$\frac{d\Theta(t)}{dt} = \left[\frac{d[X^{-1}(t)]'}{dt}\right]\{\cdots\}^{-1}X^{-1}(t) +$$

$$+ [X^{-1}(t)]' \left[\frac{d}{dt}\{\cdots\}^{-1}\right] X^{-1}(t) + [X^{-1}(t)]'\{\cdots\}^{-1} \ \frac{dX^{-1}(t)}{dt} =$$
$$= -A'(t)\Theta(t) + [X^{-1}(t)]'\{\cdots\}^{-1}X^{-1}(t)D^{-1}[X^{-1}(t)]'\{\cdots\}^{-1}X^{-1}(t) - \Theta(t)A(t) =$$
$$= -A'(t)\Theta(t) + \Theta(t)D^{-1}\Theta(t) - \Theta(t)A(t).$$

The proof of Proposition 2.1 is concluded by the two chains of implications

$$[D < 0] \Rightarrow [D^{-1} < 0] \Rightarrow \left[X^{-1}(\tau)D^{-1}[X^{-1}(\tau)]' < 0 \ \forall\tau \in [0,\vartheta]\right] \Rightarrow$$
$$\Rightarrow \left[\int_t^{\vartheta} X^{-1}(\tau)D^{-1}[X^{-1}(\tau)]'\, d\tau \leqslant 0 \quad \forall t \in [0,\vartheta]\right],$$
$$\left[C^{-1} < 0 \wedge \int_t^{\vartheta} X^{-1}(\tau)D^{-1}[X^{-1}(\tau)]'\, d\tau \leqslant 0 \quad \forall t \in [0,\vartheta]\right] \Rightarrow$$
$$\Rightarrow \left[C^{-1} + \int_t^{\vartheta} X^{-1}(\tau)D^{-1}[X^{-1}(\tau)]'\, d\tau < 0\right].$$

□

Remark 2.1. Equation (2.1) appears if a saddle point $U^0 = (U_1^0, U_2^0) \in \mathbf{U}$ is designed using dynamic programming:

$$J(U_1, U_2^0, t_0, x_0) \leqslant J(U_1^0, U_2^0, t_0, x_0) \leqslant J(U_1^0, U_2, t_0, x_0),$$

$\forall(t_0, x_0) \in [0,\vartheta) \times \mathbb{R}^n$, $U_i \in \mathbf{U}_i$ $(i = 1, 2)$, in the zero-sum two-player modification of the game Γ_2 (i.e., the game Γ_2 with $C = C_1 = -C_2$, $D = D_{11} = -D_{22}$, $D_{12} = D_{21} = O_{n\times n}$ and $J = J_1 = -J_2$). There exist several different types of the solution $\Theta(t)$, $t \in [0,\vartheta]$, of equation (2.1), that are reducible to each other. (Recall that the solution $\Theta(t)$ is nonunique.) We have selected (2.2) due to its convenience for the small parameter method.

Proposition 2.2. *Let $A(\cdot), B(\cdot) \in C^{n\times n}[0,\vartheta]$. Then the solution of the matrix differential equation*

$$\dot\Theta + \Theta A(t) + A'(t)\Theta + B(t) = O_{n\times n}, \quad \Theta(\vartheta) = C, \tag{2.5}$$

has the form

$$\Theta(t) = [X^{-1}(t)]' \left\{C + \int_t^{\vartheta} X'(\tau)B(\tau)X(\tau)\, d\tau\right\} X^{-1}(t), \tag{2.6}$$

where $X(t)$ is the fundamental matrix of solutions for the system

$$\dot X = A(t)X, \quad X(\vartheta) = E_n.$$

P r o o f. The matrix system (2.5) is linear in x, inhomogeneous and also consists of continuous in $t \in [0,\vartheta]$ coefficients. For any $t_0 \in [0,\vartheta)$, such a system has a unique continuous differentiable solution $\Theta(t)$ that is extendable to the interval $[0,\vartheta]$.

Finally, we will demonstrate that $\Theta(t)$ is given by (2.6). Really,

$$[X(\vartheta) = E_n] \Rightarrow [\det X(t) \neq 0 \ \forall t \in [0,\vartheta]] \Rightarrow \left[\exists X^{-1}(t) \ \forall t \in [0,\vartheta]\right].$$

In view of (2.4), differentiating both sides of (2.6) yields

$$\frac{d\Theta(t)}{dt} = \left[\frac{d[X^{-1}(t)]'}{dt}\right]\{\cdots\}X^{-1}(t) + [X^{-1}(t)]'\left[\frac{d}{dt}\{\cdots\}\right]X^{-1}(t) +$$
$$+ [X^{-1}(t)]'\{\cdots\}\frac{dX^{-1}(t)}{dt} = -A'(t)\Theta(t) - B(t) - \Theta(t)A(t).$$

From (2.6) it follows that, at $t = \vartheta$, $\Theta(\vartheta) = E_nCE_n = C$. □

§ 3. No maxima in Γ_2

The next result can be used to eliminate the linear-quadratic differential games Γ_2 without any Berge and/or Nash equilibrium, depending on the sign definiteness of the quadratic forms appearing in the integrand of the payoff functions (1.2) of the players.

Lemma 3.1. *Let the quadratic form $u_1' D_{11} u_1$ in (1.2) be positive definite. For any strategy profile $U^* = (U_1^*, U_2^*) \in \boldsymbol{U}$, where $U_i^* \div Q_i^*(t)x$ $(i = 1, 2)$ and $Q_i^*(\cdot) \in C^{n\times n}[0, \vartheta]$, any initial position $(t_0, x_0) \in [0, \vartheta) \times \mathbb{R}^n$, $x_0 \neq 0_n$, and any constant and symmetric matrices C_1 and D_{12} there exists a strategy $\widetilde{U}_1 \in \boldsymbol{U}_1$, $\widetilde{U}_1 \div \widetilde{Q}_1(t)x$, of player 1 such that*

$$J_1(\widetilde{U}_1, U_2^*) > J_1(U_1^*, U_2^*). \tag{3.1}$$

P r o o f. Consider some frozen strategy profile from $\mathbf{U}$,

$$U^* = (U_1^*, U_2^*) \div (Q_1^*(t)x, Q_2^*(t)x)\,, \quad Q_i^*(\cdot) \in C^{n\times n}[0, \vartheta] \quad (i = 1, 2),$$

and also some frozen initial position $(t_0, x_0) \in [0, \vartheta) \times [\mathbb{R}^n \backslash \{0_n\}]$.

The proof of Lemma 3.1 includes two stages as follows. In the first stage, we will establish the existence of a quadratic form $V(t, x) = x'\Theta(t)x$ for which

$$J_1(U_1^*, U_2^*, t_0, x_0) = V(t_0, x_0).$$

In the second stage, we will find a strategy $\widetilde{U}_1 \in \mathbf{U}_1$ of player 1 that satisfies (3.1).

First stage. Following the dynamic programming method we construct the scalar function

$$W(t, x, u_1, u_2, V) = \frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial x}\right]' (A(t)x + u_1 + \varepsilon u_2) + u_1' D_{11} u_1 + u_2' D_{12} u_2. \tag{3.2}$$

For $u_i = Q_i^*(t)x$ $(i = 1, 2)$,

$$W[t, x, V] = W(t, x, u_1 = Q_1^*(t)x, u_2 = Q_2^*(t)x, V) =$$

$$= \frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial x}\right]' (A(t)x + Q_1^*(t)x + \varepsilon Q_2^*(t)x) + [Q_1^*(t)x]' D_{11} Q_1^*(t)x + [Q_2^*(t)x]' D_{12} Q_2^*(t)x.$$

Next, we solve the partial differential equation

$$W[t, x, V] = 0, \quad V(\vartheta, x) = x'C_1 x. \tag{3.3}$$

The solution $V = V(t, x)$ is constructed in the class of the quadratic forms $V(t, x) = x'\Theta(t)x$ with a continuously differentiable (in t) matrix $\Theta(t)$ of dimensions $n \times n$, and this fact will be indicated by $\Theta(\cdot) \in C^1_{n\times n}[0, \vartheta]$.

Substituting $V(t, x) = x'\Theta(t)x$ into (3.3) and collecting similar terms at the n-dimensional vector $x \in \mathbb{R}^n$ give

$$W[t, x, V(t, x) = x'\Theta(t)x] = x' \left\{ \frac{d\Theta(t)}{dt} + [\Theta(t)]'[A(t) + Q_1^*(t) + \varepsilon Q_2^*(t)] + \right.$$

$$\left. + [A'(t) + (Q_1^*(t))' + \varepsilon (Q_2^*(t))']\Theta(t) + [Q_1^*(t)]' D_{11} Q_1^*(t) + [Q_2^*(t)]' D_{12} Q_2^*(t) \right\} x = 0,$$

$$x'\Theta(\vartheta)x = x'C_1 x.$$

Both of these identities will hold if, for all $t \in [0, \vartheta]$ the matrix $\Theta(t)$ of dimensions $n \times n$ is the solution of the linear inhomogeneous matrix differential equation

$$\dot{\Theta} + \Theta'[A(t) + Q_1^*(t) + \varepsilon Q_2^*(t)] + [A'(t) + (Q_1^*(t))' + \varepsilon(Q_2^*(t))']\Theta + B(t) = O_{n\times n} \tag{3.4}$$

with continuous in t elements and the boundary-value condition

$$\Theta(\vartheta) = C_1, \tag{3.5}$$

where the matrix

$$B(t) = [Q_1^*(t)]' D_{11} Q_1^*(t) + [Q_2^*(t)]' D_{12} Q_2^*(t) \tag{3.6}$$

is continuous and symmetric.

By Proposition 2.2, the system (3.4), (3.5) has a unique continuously differentiable solution $\Theta = \Theta^*(t)$ that is extendable to any interval $[t_0, \vartheta] \subset [0, \vartheta]$. Due to the symmetric property of the matrices C and $B(t)$ from (3.6) and the explicit form (2.6) of $\Theta^*(t)$, the matrix $\Theta^*(t)$ will be symmetric for all $t \in [t, \vartheta]$.

Now, we will construct the realizations of the frozen strategies $U_i^* \div u_i^*(t,x) = Q_i^*(t)x$ along the solution $x^*(t)$ to the vector equation (1.1), i.e., we will construct $u_i^*[t] = Q_i^*(t)x^*(t)$, $t \in [t_0, \vartheta]$ $(i = 1, 2)$, where

$$\frac{dx^*(t)}{dt} = A(t)x^*(t) + Q_1^*(t)x^*(t) + \varepsilon Q_2^*(t)x^*(t), \quad x^*(t_0) = x_0.$$

In view of (3.3), it follows that

$$W[t, x^*(t), V(t, x^*(t)) = [x^*(t)]'\Theta^*(t)x^*(t)] = W^*[t] = 0 \tag{3.7}$$

for all $t \in [t_0, \vartheta]$ along the solution of (3.4), (3.5) and (1.1). Due to (3.5), we have $V(\vartheta, x^*(\vartheta)) = [x^*(\vartheta)]'C_1x^*(\vartheta)$; then integrating both sides of (3.7) from t_0 to ϑ gives

$$\begin{aligned}
0 = \int_{t_0}^{\vartheta} W^*[t]\,dt = \int_{t_0}^{\vartheta} \Bigg\{ \frac{\partial V(t,x)}{\partial t} + \left[\frac{\partial V(t,x)}{\partial x}\right]' [A(t)x + Q_1^*(t)x + \varepsilon Q_2^*(t)x] + \\
+ [Q_1^*(t)]' D_{11} Q_1^*(t) + [Q_2^*(t)]' D_{12} Q_2^*(t) \Bigg\}\Bigg|_{x = x^*(t)} dt = \\
= \int_{t_0}^{\vartheta} \frac{dV^*(t, x^*(t))}{dt}\,dt + \int_{t_0}^{\vartheta} \{(u_1^*[t])' D_{11} u_1^*[t] + (u_2^*[t])' D_{12} u_2^*[t]\}\,dt = \\
= V(\vartheta, x^*(\vartheta)) - V(t_0, x_0) + \int_{t_0}^{\vartheta} \{(u_1^*[t])' D_{11} u_1^*[t] + (u_2^*[t])' D_{12} u_2^*[t]\}\,dt = \\
= [x^*(\vartheta)]' C_1 x^*(\vartheta) + \int_{t_0}^{\vartheta} \{(u_1^*[t])' D_{11} u_1^*[t] + (u_2^*[t])' D_{12} u_2^*[t]\}\,dt - V(t_0, x_0) = \\
= J_1(U_1^*, U_2^*, t_0, x_0) - V(t_0, x_0).
\end{aligned}$$

This directly leads to the equality

$$V(t_0, x_0) = x_0' \Theta^*(t_0) x_0 = J_1(U^*, t_0, x_0).$$

Second stage. Consider the strategy $\widetilde{U}_1 \div \widetilde{u}_1(t,x) = \beta x$ of player 1, where a numerical parameter $\beta > 0$ will be determined below. Due to the symmetry of the matrix D_{11} and the condition $D_{11} > 0$,

$$u_1' D_{11} u_1 \geqslant \lambda_1 \|u_1\|^2 = \lambda_1 u_1' u_1 \quad \forall u_1 \in \mathbb{R}^n. \tag{3.8}$$

Here $\|\cdot\|$ denotes the Euclidean norm and $\lambda_1 > 0$ is the smallest root of the characteristic equation $\det[D_{11} - \lambda E_n] = 0$ [9, pp. 88, 109]; E_n denotes the identity matrix of dimentions $n \times n$.

We will adopt the matrix $\Theta^*(t)$, $t \in [0,\vartheta]$, of dimensions $n \times n$ obtained in the first stage of solving the problem (3.4), (3.5). (Note that the elements $\Theta^*(t)$ are continuously differentiable with respect to t). Taking inequality (3.8) into account, also we will use the strategy $U_2^* \div Q_2^*(t)x$ of player 2 chosen in the first stage.

In view of (3.8), following (3.2) we construct the function

$$\begin{aligned}
\widetilde{W}[t,x] &= W(t,x,\widetilde{u}_1(t,x) = \beta x, u_2^*(t,x) = Q_2^*(t)x, V(t,x) = x'\Theta^*(t)x) = \\
&= \frac{\partial V(t,x)}{\partial t} + \left[\frac{\partial V(t,x)}{\partial x}\right]' [A(t)x + \widetilde{u}_1(t,x) + \varepsilon u_2^*(t,x)] + \\
&\quad + [\widetilde{u}_1(t,x)]' D_{11} \widetilde{u}_1(t,x) + [u_2^*(t,x)]' D_{12} u_2^*(t,x) \geqslant \\
&\geqslant x' \frac{d\Theta^*(t)}{dt} x + 2x'\Theta^*(t)[A(t) + \beta E_n + \varepsilon Q_2^*(t)]x + x'\lambda_1\beta^2 E_n x + x'[Q_2^*(t)]' D_{12} Q_2^*(t) x = \\
&= x' \left\{ \frac{d\Theta^*(t)}{dt} + \Theta^*(t)[A(t) + \beta E_n + \varepsilon Q_2^*(t)] + \big[A'(t) + \beta E_n + \varepsilon [Q_2^*(t)]'\big]\Theta^*(t) + \right. \\
&\quad \left. + \lambda_1 \beta^2 E_n + [Q_2^*(t)]' D_{12} Q_2^*(t) \right\} x = x' M(t,\beta) x.
\end{aligned}$$

The parenthesized matrix $M(t,\beta)$ of dimensions $n \times n$ is symmetric and has the form

$$M(t,\beta) = \lambda_1 \beta^2 E_n + 2\beta\Theta^*(t) + K(t),$$

with the matrix

$$K(t) = \dot{\Theta}^*(t) + \Theta^*(t)[A(t) + \varepsilon\Theta_2^*(t)] + [A(t) + \varepsilon\Theta_2^*(t)]'\Theta^*(t) + [Q_2^*(t)]' D_{12} Q_2^*(t)$$

of dimensions $n \times n$.

The elements of the matrix $M(t,\beta)$ are continuous in $t \in [0,\vartheta]$ and hence uniformly bounded on the compact set $[0,\vartheta]$. The factor β^2 appears in the diagonal elements of the matrix $M(t,\beta)$ only. As before, $\lambda_1 > 0$ is the smallest root of the characteristic equation $\det[D_{11} - \lambda E_n] = 0$.

Therefore, the constant $\beta = \beta(U_1^*) > 0$ can be chosen sufficiently large so that all leading minors of the matrix $M(t,\beta)$ become positive for all $t \in [0,\vartheta]$ and for all $\beta > \beta(U_1^*)$. By Silvester's criterion [9, p. 88], the quadratic form $x'M(t,\beta)x$ is positive definite for all $t \in [0,\vartheta]$ and constants $\beta > \beta(U_1^*)$ because the sign of $x'M(t,\beta)x$ is determined by the sign of the quadratic form $\beta^2\lambda_1 x'x$.

We fix some constant $\beta^* > \beta(U_1^*)$; then

$$\widetilde{W}[t,x] \geqslant x'M(t,\beta^*)x > 0 \quad \forall t \in [0,\vartheta] \quad \forall x \in \mathbb{R}^n \setminus \{0_n\}. \tag{3.9}$$

Denote by $\widetilde{x}(t)$, $t \in [0,\vartheta]$, the solution of the vector equation

$$\dot{x} = A(t)x + \beta^* E_n x + \varepsilon Q_2^*(t)x, \quad x(t_0) = x_0.$$

Since $[x_0 \neq 0_n] \Rightarrow [\widetilde{x}(t) \neq 0_n \ \forall t \in [t_0, \vartheta]]$, according to (3.9) we have

$$\widetilde{W}[t, \widetilde{x}(t)] > 0 \quad \forall t \in [t_0, \vartheta].$$

Integrating both sides of this inequality from t_0 to ϑ and using the boundary-value condition $\Theta^*(\vartheta) = C_1$ from (3.5) and also $\widetilde{u}_1^*[t] = \beta \widetilde{x}(t)$ we obtain

$$0 < \int_{t_0}^{\vartheta} \widetilde{W}[t, \widetilde{x}(t)]\, dt = \int_{t_0}^{\vartheta} \left\{ \frac{\partial V(t,x)}{\partial t} + \left[\frac{\partial V(t,x)}{\partial x}\right]' [A(t)x + \beta^* E_n x + \varepsilon Q_2^*(t)x] \right\}\bigg|_{x=\widetilde{x}(t)} dt +$$

$$+ \int_{t_0}^{\vartheta} \{x'\beta^* D_{11} \beta^* x + [Q_2^*(t)]' D_{12} Q_2^*(t) x\}\Big|_{x=\widetilde{x}(t)} dt =$$

$$= \int_{t_0}^{\vartheta} \left\{ \frac{dV(t, \widetilde{x}(t))}{dt} \right\} dt + \int_{t_0}^{\vartheta} \{(\widetilde{u}_1^*[t])' D_{11} \widetilde{u}_1^*[t] + (u_2^*[t])' D_{12} u_2^*[t]\}\, dt =$$

$$= \widetilde{x}'(\vartheta) C_1 \widetilde{x}(\vartheta) + \int_{t_0}^{\vartheta} \{(\widetilde{u}_1^*[t])' D_{11} \widetilde{u}_1^*[t] + (u_2^*[t])' D_{12} u_2^*[t]\}\, dt - V(t_0, x_0) =$$

$$= J_1(\widetilde{U}_1, U_2^*, t_0, x_0) - V(t_0, x_0).$$

In combination with $J_1(U_1^*, U_2^*, t_0, x_0) = V(t_0, x_0)$ this result finally proves Lemma 3.1. □

Remark 3.1. Consider the inner optimization problem in the game Γ_2: find $\max\limits_{U_1 \in \mathbf{U}_1} J_1(U_1, U_2^*, t_0, x_0)$ subject to the constraint (1.1) with a fixed strategy $U_2^* \in \mathbf{U}_2$ of player 2 and any $(t_0, x_0) \in \in [0, \vartheta] \times [\mathbb{R}^n \setminus \{0_n\}]$. In fact, Lemma 3.1 states that this maximization problem has no solution for $D_{11} > 0$ and $x_0 \neq 0_n$. Indeed, whatever the strategy $U_1^* \in \mathbf{U}_1$ of player 1 is, there always exists a strategy $\widetilde{U}_1 \in \mathbf{U}_1$ such that

$$J_1(\widetilde{U}_1, U_2^*) > J_1(U_1^*, U_2^*)$$

for all $(t_0, x_0) \in [0, \vartheta) \times [\mathbb{R}^n \setminus \{0_n\}]$. This result can be used for eliminating the solution concepts of the game Γ_2 that maximize the payoff function of player 1 (e.g., avoiding Nash equilibrium in the game Γ_2 with $D_{11} > 0$). By analogy with Lemma 3.1, we may demonstrate that the game Γ_2 with $D_{12} > 0$ has no Berge equilibrium and hence the players should not choose this solution principle for the game Γ_2 with $D_{12} > 0$.

§ 4. Formalization of equilibria and sufficient conditions

Definition 4.1. A strategy profile $U^e = (U_1^e, U_2^e) \in \mathbf{U}$ is a Nash equilibrium in the game Γ_2 if, for any initial position $(t_0, x_0) \in [0, \vartheta) \times [\mathbb{R}^n \setminus \{0_n\}]$,

$$\max_{U_1 \in \mathbf{U}_1} J_1(U_1, U_2^e, t_0, x_0) = J_1(U^e, t_0, x_0),$$
$$\max_{U_2 \in \mathbf{U}_2} J_2(U_1^e, U_2, t_0, x_0) = J_2(U^e, t_0, x_0).$$

Definition 4.2. A strategy profile $U^B = (U_1^B, U_2^B) \in \mathbf{U}$ is a Berge equilibrium in the game Γ_2 if, for any initial position $(t_0, x_0) \in [0, \vartheta) \times [\mathbb{R}^n \setminus \{0_n\}]$,

$$\max_{U_2 \in \mathbf{U2}} J_1(U_1^B, U_2, t_0, x_0) = J_1(U^B, t_0, x_0),$$
$$\max_{U_1 \in \mathbf{U}_1} J_2(U_1, U_2^B, t_0, x_0) = J_2(U^B, t_0, x_0).$$

Remark 4.1. Despite the seeming similarity of these two types of equilibria, they have a deep distinction as follows. Unlike Definition 4.1 expressing the selfish character of each player (maximization of his own payoff), Definition 4.2 postulates altruism, guiding each player towards *the Golden Rule of ethics* — "behave unto the opponent as you would like him to behave unto you."

The *sufficient conditions* that guarantee the existence of a Nash equilibrium and a Berge equilibrium in the linear-quadratic game under study (see below) are the result of applying dynamic programming to Definitions 4.1 and 4.2 respectively. They were derived in the book [11, pp. 112, 124].

First, we introduce the two scalar functions

$$W_i(t,x,u_1,u_2,V) = \frac{\partial V_i}{\partial t} + \left[\frac{\partial V_i}{\partial x}\right]'[A(t)x + u_1 + \varepsilon u_2] + u_1'D_{i1}u_1 + u_2'D_{i2}u_2 \ (i=1,2), \quad (4.1)$$

where $V = (V_1, V_2) \in \mathbb{R}^2$.

Nash equilibrium

Proposition 4.1. *Let $V_i^e(t,x)$ $(i=1,2)$ be unique continuously differentiable scalar functions such that*

1^0)

$$V_i^e(\vartheta, x) = x'C_i x \quad \forall x \in \mathbb{R}^n \quad (4.2)$$

2^0) *Let $u_i^e(t,x,V^e)$ $(i=1,2)$ be vector functions such that*

$$\begin{aligned} &\max_{u_1}\{W_1(t,x,u_1,u_2^e(t,x,V^e),V^e)\} = Idem\,\{u_1 \to u_1^e(t,x,V^e)\}, \\ &\max_{u_2}\{W_2(t,x,u_1^e(t,x,V^e),u_2,V^e)\} = Idem\,\{u_2 \to u_2^e(t,x,V^e)\} \end{aligned} \quad (4.3)$$

for all $(t,x) \in [0,\vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$ and $V^e = (V_1^e, V_2^e) \in \mathbb{R}^2$.

3^0) *Let the functions $V_i^e(t,x)$ $(i=1,2)$ be the solution for the system of two partial differential equations*

$$W_i(t,x,u_1^e(t,x,V^e),u_2^e(t,x,V^e),V^e) = 0 \quad (i=1,2) \quad (4.4)$$

with the boundary value conditions (4.2) for all $(t,x) \in [0,\vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$.

4^0) *Let strategies $U_i^e \div u_i^e(t,x,V^e(t,x)) = u_i^e[t,x]$ be such that $U_i^e \in \boldsymbol{U}_i$ $(i=1,2)$.*

Then

a) *the strategy profile $U^e = (U_1^e, U_2^e)$ is a Nash equilibrium in the game Γ_2 (in terms of Definition 4.1)*

b) *the Nash equilibrium payoffs are*

$$J_i(U^e, t_0, x_0) = V_i^e(t_0, x_0) \quad (i=1,2).$$

Remark 4.2. In practice, a Nash equilibrium should be designed by constructing the scalar functions $W_i(t, x, u_1, u_2, V^e)$ (4.1) and proceeding with items 1^0)–4^0) of Proposition 4.1. More specifically, letting $V_i^e(t, x) = x'\Theta_i^e(t)x$, $[\Theta_i^e(t)]' = \Theta_i^e(t)$ $(i = 1, 2)$, we have to perform the following steps.

Step 1. Using (4.2), find $\Theta_i^e(\vartheta) = C_i$ $(i = 1, 2)$.

Step 2. Based on (4.3) and (1.3)–(1.5), construct $u_i^e(t, x, V^e)$ $(i = 1, 2)$.

Step 3. Find the solution $V_i^e(t, x)$ $(i = 1, 2)$ for the system of two partial differential equations (4.4) with the boundary-value conditions (4.2).

Step 4. Check that $u_i^e[t, x] = u_i(t, x, V^e(t, x)) = Q_i^e(t)x$ and $Q_i^e(\cdot) \in C^{n\times n}[0, \vartheta]$ $(i = 1, 2)$.

The resulting pair $U^e = (U_1^e, U_2^e)$ is a Nash equilibrium in the game Γ_2 and the corresponding payoffs of the players are $J_i(U^e, t_0, x_0) = V_i^e(t_0, x_0)$ $(i = 1, 2)$.

Berge equilibrium

Proposition 4.2. *Let $V_i^B(t, x)$ $(i = 1, 2)$ be unique continuously differentiable scalar functions such that*

1^0)
$$V_i^B(\vartheta, x) = x'C_ix \quad \forall x \in \mathbb{R}^n. \tag{4.5}$$

2^0) *Let $u_i^B(t, x, V^B)$ $(i = 1, 2)$ be vector functions such that*
$$\begin{aligned} &\max_{u_2}\{W_1(t, x, u_1^B(t, x, V^B), u_2, V^B)\} = Idem\,\{u_2 \to u_2^B(t, x, V^B)\},\\ &\max_{u_1}\{W_2(t, x, u_1, u_2^B(t, x, V^B), V^B)\} = Idem\,\{u_1 \to u_1^B(t, x, V^B)\} \end{aligned} \tag{4.6}$$

for all $(t, x) \in [0, \vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$ and $V^B = (V_1^B, V_2^B) \in \mathbb{R}^2$.

3^0) *Let the functions $V_i^B(t, x)$ $(i = 1, 2)$ be the solution for the system of two partial differential equations*
$$W_i(t, x, u_1^B(t, x, V^B), u_2^B(t, x, V^B), V^B) = 0 \quad (i = 1, 2) \tag{4.7}$$

with the boundary-value conditions (4.5) for all $(t, x) \in [0, \vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$.

4^0) *Let the strategies $U_i^B \div u_i^B(t, x, V^B(t, x)) = u_i^B[t, x]$ be such that $U_i^B \in \boldsymbol{U}_i$ $(i = 1, 2)$.*

Then

a) *the strategy profile $U^B = (U_1^B, U_2^B)$ is a Berge equilibrium in the game Γ_2 (in terms of Definition 4.2);*

b) *the Berge equilibrium payoffs are*
$$J_i(U^B, t_0, x_0) = V_i^B(t_0, x_0) \quad (i = 1, 2) \tag{4.8}$$

Remark 4.3. Like in the case of Nash equilibrium, a Berge equilibrium should be designed in four steps corresponding to the items 1^0) – 4^0) of Proposition 4.2. As the functions $V_i^B(t, x)$ we should choose the quadratic form $V_i^B(t, x) = x'\Theta_i^B(t)x$, where $[\Theta_i^B(t)]' = \Theta_i^B(t)$ for all $t \in [0, \vartheta]$ $(i = 1, 2)$.

§ 5. Explicit form of equilibria

Nash equilibrium

Proposition 5.1. *Consider the game Γ_2 with the matrices*

$$D_{11} < 0, \quad D_{22} < 0, \quad C_1 < 0. \tag{5.1}$$

If the system of Riccati matrix equations

$$\begin{cases} \dot{\Theta}_1^e + \Theta_1^e A(t) + A'(t)\Theta_1^e - \Theta_1^e D_{11}\Theta_1^e - \varepsilon^2[\Theta_1^e D_{22}^{-1}\Theta_2^e + \Theta_2^e D_{22}^{-1}\Theta_1^e] - \\ -\varepsilon^2\Theta_2^e D_{22}^{-1} D_{12} D_{22}^{-1}\Theta_2^e = O_{n\times n}, \quad \Theta_1^e(\vartheta,\varepsilon) = C_1, \\ \dot{\Theta}_2^e + \Theta_2^e[A(t) - D_{11}^{-1}\Theta_1^e] + [A'(t) - \Theta_1^e D_{11}^{-1}]\Theta_2^e + \\ +\Theta_1^e D_{11}^{-1} D_{12} D_{11}^{-1}\Theta_1^e - \varepsilon^2\Theta_2^e D_{22}^{-1}\Theta_2^e = O_{n\times n}, \quad \Theta_2^e(\vartheta,\varepsilon) = C_2, \end{cases} \tag{5.2}$$

has a solution $(\Theta_1^e(t,\varepsilon), \Theta_2^e(t,\varepsilon))$ that is extendable to $[0,\vartheta]$, then in the game Γ_2

a) *the Nash equilibrium is given by*

$$U^e = (U_1^e, U_2^e) \div (-D_{11}^{-1}\Theta_1^e(t,\varepsilon)x, -\varepsilon D_{22}^{-1}\Theta_2^e(t,\varepsilon)x); \tag{5.3}$$

b) *the Nash equilibrium payoffs of the players are*

$$J_i(U^e, t_0, x_0) = x_0'\Theta_i^e(t_0,\varepsilon)x_0 \ \ (i = 1,2). \tag{5.4}$$

P r o o f. Following Remark 4.2 we construct the functions

$$W_i^e(t,x,u_1,u_2,V) = \frac{\partial V_i}{\partial t} + \left[\frac{\partial V_i}{\partial x}\right]'[A(t)x + u_1 + \varepsilon u_2] + u_1' D_{i1} u_1 + u_2' D_{i2} u_2 \ (i = 1,2). \tag{5.5}$$

Step 1. In view of (4.2) and $V_i^e(t,x) = x'\Theta_i^e(t)x$,

$$V_i^e(\vartheta,x) = x'\Theta_i^e(\vartheta,\varepsilon)x = x'C_i x \ \ \forall x \in \mathbb{R}^n \setminus \{0_n\},$$

which gives

$$\Theta_i^e(\vartheta,\varepsilon) = C_i \ (i = 1,2). \tag{5.6}$$

Step 2. Due to (4.3),

$$\max_{u_1}\{W_1(t,x,u_1,u_2^e(t,x,V^e),V^e)\} = Idem\,\{u_1 \to u_1^e(t,x,V^e)\}.$$

This equality holds if, according to (1.3)–(1.5),

$$\left.\frac{\partial W_1(t,x,u_1,u_2^e(t,x,V^e),V^e)}{\partial u_1}\right|_{u_1(t,x,V^e)} = \frac{\partial V_1^e}{\partial x} + 2D_{11}u_1^e(t,x,V^e) = 0_n,$$

$$\left.\frac{\partial^2 W_1(t,x,u_1,u_2^e(t,x,V^e),V^e)}{\partial u_1^2}\right|_{u_1(t,x,V^e)} = 2D_{11} < 0,$$

for any $(t,x) \in [0,\vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$ and $V^e = (V_1^e, V_2^e) \in \mathbb{R}^2$. By analogy,

$$\left.\frac{\partial W_2(t,x,u_1^e(t,x,V^e),u_2,V^e)}{\partial u_2}\right|_{u_2(t,x,V^e)} = \varepsilon\frac{\partial V_2^e}{\partial x} + 2D_{22}u_2^e(t,x,V^e) = 0_n,$$

$$\left.\frac{\partial^2 W_2(t,x,u_1^e(t,x,V^e),u_2,V^e)}{\partial u_2^2}\right|_{u_2(t,x,V^e)} = 2D_{22} < 0,$$

for all $(t, x) \in [0, \vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$ and $V^e = (V_1^e, V_2^e) \in \mathbb{R}^2$.

The second and fourth relations are true by (5.1). Using the first and third relations, we find

$$u_1^e(t, x, V^e) = -\frac{1}{2} D_{11}^{-1} \frac{\partial V_1^e}{\partial x}, \quad u_2^e(t, x, V^e) = -\frac{\varepsilon}{2} D_{22}^{-1} \frac{\partial V_2^e}{\partial x}. \tag{5.7}$$

Step 3. We write the two partial differential equations (4.4), with the boundary-value conditions (5.6) to find the two scalar functions $V_i^e(t, x)$ $(i = 1, 2)$:

$$\begin{aligned}
&0 = W_1^e[t, x, V^e] = W_1(t, x, u_1^e(t, x, V^e), u_2^e(t, x, V^e), V^e) = \\
&= \frac{\partial V_1^e}{\partial t} + \left[\frac{\partial V_1^e}{\partial x}\right]' \left[A(t)x - \frac{1}{2} D_{11}^{-1} \frac{\partial V_1^e}{\partial x}\right] - \frac{\varepsilon^2}{2} \left[\frac{\partial V_1^e}{\partial x}\right]' D_{22}^{-1} \frac{\partial V_2^e}{\partial x} + \\
&+ \frac{1}{4} \left[\frac{\partial V_1^e}{\partial x}\right]' D_{11}^{-1} \frac{\partial V_1^e}{\partial x} + \frac{\varepsilon^2}{4} \left[\frac{\partial V_2^e}{\partial x}\right]' D_{22}^{-1} D_{12} D_{22}^{-1} \frac{\partial V_2^e}{\partial x} = \\
&= \frac{\partial V_1^e}{\partial t} + \left[\frac{\partial V_1^e}{\partial x}\right]' A(t)x - \frac{\varepsilon^2}{2} \left[\frac{\partial V_1^e}{\partial x}\right]' D_{22}^{-1} \frac{\partial V_2^e}{\partial x} - \\
&- \frac{1}{4} \left[\frac{\partial V_1^e}{\partial x}\right]' D_{11}^{-1} \frac{\partial V_1^e}{\partial x} + \frac{\varepsilon^2}{4} \left[\frac{\partial V_2^e}{\partial x}\right]' D_{22}^{-1} D_{12} D_{22}^{-1} \frac{\partial V_2^e}{\partial x}, \\
&W_2^e[t, x, V^e] = W_2(t, x, u_1^e(t, x, V^e), u_2^e(t, x, V^e), V^e) = \\
&= \frac{\partial V_2^e}{\partial t} + \left[\frac{\partial V_2^e}{\partial x}\right]' \left[A(t)x - \frac{1}{2} D_{11}^{-1} \frac{\partial V_1^e}{\partial x} - \frac{\varepsilon^2}{2} D_{22}^{-1} \frac{\partial V_2^e}{\partial x}\right] + \\
&+ \frac{1}{4} \left[\frac{\partial V_1^e}{\partial x}\right]' D_{11}^{-1} D_{21} D_{11}^{-1} \frac{\partial V_1^e}{\partial x} - \frac{\varepsilon^2}{4} \left[\frac{\partial V_2^e}{\partial x}\right]' D_{22}^{-1} \frac{\partial V_2^e}{\partial x} = 0.
\end{aligned} \tag{5.8}$$

In view of (1.3) and $V_i^e(t, x) = x' \Theta_i^e(t) x$, we obtain the gradients $\frac{\partial V_i^e}{\partial x} = 2\Theta_i^e(t)x$ and $\frac{\partial V_i^e}{\partial t} = x' \frac{d\Theta_i^e}{dt} x$. Substituting $\frac{\partial V_i^e}{\partial x}$ and $\frac{\partial V_i^e}{\partial t}$ into (5.8) and collecting like terms with the pairwise products of the components of the n-dimensional vector x, we arrive at the equations

$$\begin{aligned}
W_1^e[t, x, V^e] = x' \Big\{ &\frac{d\Theta_1^e}{dt} + \Theta_1^e A(t) + A'(t) \Theta_1^e - \Theta_1^e D_{11}^{-1} \Theta_1^e + \\
&+ \varepsilon^2 \big[- \Theta_1^e D_{22}^{-1} \Theta_2^e - \Theta_2^e D_{22}^{-1} \Theta_1^e + \Theta_2^e D_{22}^{-1} D_{12} D_{22}^{-1} \Theta_2^e \big] \Big\} x = 0, \\
W_2^e[t, x, V^e] = x' \Big\{ &\frac{d\Theta_2^e}{dt} + \Theta_2^e A(t) + A'(t) \Theta_2^e - \Theta_1^e D_{11}^{-1} D_{21} D_{11}^{-1} \Theta_1^e - \\
&- \Theta_2^e D_{11}^{-1} \Theta_1^e - \varepsilon^2 \Theta_2^e D_{22}^{-1} \Theta_2^e \Big\} x = 0
\end{aligned}$$

with the boundary-value conditions

$$V_i(\vartheta, x) = x' \Theta_i^e(\vartheta, \varepsilon) x = x' C_i x \quad (i = 1, 2).$$

The identities $W_i^e[t, x, V^e(t, x)] = 0$ for all $(t, x) \in [0, \vartheta] \times [\mathbb{R}^n \backslash \{0_n\}]$ $(i = 1, 2)$ hold if the system of Riccati matrix equations (5.2) has a solution $(\Theta_1^e(t, \varepsilon), \Theta_2^e(t, \varepsilon))$ that is extendable to the interval $[0, \vartheta]$. The condition $C_1 < 0$ ensures the existence of a solution of the first equation from (5.2) for $\varepsilon = 0$.

Step 4. Using the obtained solutions $(\Theta_1^e(t,\varepsilon), \Theta_2^e(t,\varepsilon))$ and (5.7) we find the two n-dimensional vector functions

$$u_1^e[t,x] = u_1(t,x,V^e(t,x)) = -D_{11}^{-1}\Theta_1^e(t,\varepsilon)x,$$
$$u_2^e[t,x] = u_2(t,x,V^e(t,x)) = -\varepsilon D_{22}^{-1}\Theta_2^e(t,\varepsilon)x.$$

Since $D_{11}^{-1}\Theta_1^e(\cdot,\varepsilon)$, $\varepsilon D_{22}^{-1}\Theta_2^e(\cdot,\varepsilon) \in C^1_{n\times n}[0,\vartheta]$, the Nash equilibrium in the game Γ_2 will have the form (5.3), and the Nash equilibrium payoffs of the players will be given by (5.4). □

Remark 5.1. In the case $D_{11} > 0$ and/or $D_{22} > 0$, by Lemma 3.1 at least one of the two maxima from Definition 4.1 is not achieved for any $x_0 \neq 0_n$. Really, assume on the contrary that, e.g., in the case $D_{11} > 0$ there exists a strategy $\widehat{U}_1 \in \mathbf{U}_1$ of player 1 such that, for $x_0 \neq 0_n$,

$$\max_{U_1\in\mathbf{U}_1} J_1(U_1, U_2^e, t_0, x_0) = J_1(\widehat{U}_1, U_2^e, t_0, x_0).$$

Then, according to Lemma 3.1, for the initial position $(t_0, x_0) \in [0,\vartheta) \times [\mathbb{R}^n \backslash \{0_n\}]$ there also exists a strategy $\widetilde{U}_1 \in \mathbf{U}_1$ for which

$$J_1(\widetilde{U}_1, U_2^*) > J_1(\widehat{U}_1^*, U_2^*),$$

which contradicts the whole essence of the operator $\max\limits_{U_1\in\mathbf{U}_1}$. Thus, we have established the following result: if $D_{11} > 0$ and/or $D_{22} > 0$, then there exists no Nash equilibrium in the game Γ_2 for any initial position $(t_0, x_0) \in [0,\vartheta) \times [\mathbb{R}^n \backslash \{0_n\}]$).

Berge equilibrium

We will utilize Remark 4.3, repeating Steps 1–4 from Remark 4.2, with appropriate modifications dictated by Proposition 4.2.

Proposition 5.2. *Consider the game Γ_2 with the matrices*

$$D_{12} < 0, \quad D_{21} < 0, \quad C_2 < 0. \tag{5.9}$$

If the system of Riccati matrix equations

$$\begin{cases} \dot{\Theta}_1^B + \Theta_1^B[A(t) - D_{21}^{-1}\Theta_2^B] + [A'(t) - \Theta_2^B D_{21}^{-1}]\Theta_1^B + \\ \quad + \Theta_2^B D_{21}^{-1} D_{11} D_{21}^{-1}\Theta_2^B - \varepsilon^2 \Theta_1^B D_{12}^{-1}\Theta_1^B = O_{n\times n}, \quad \Theta_1^B(\vartheta,\varepsilon) = C_1, \\ \dot{\Theta}_2^B + \Theta_2^B A(t) + A'(t)\Theta_2^B - \Theta_2^B D_{21}\Theta_2^B + \varepsilon^2[-\Theta_2^B D_{12}^{-1}\Theta_1^B - \Theta_1^B D_{12}^{-1}\Theta_2^B + \\ \quad + \Theta_1^B D_{12}^{-1} D_{22} D_{12}^{-1}\Theta_1^B] = O_{n\times n}, \quad \Theta_2^B(\vartheta,\varepsilon) = C_2, \end{cases} \tag{5.10}$$

has a solution $(\Theta_1^B(t,\ \varepsilon), \Theta_2^B(t,\ \varepsilon))$ that is extendable to $[0,\vartheta]$, then in the game Γ_2

a) *the Berge equilibrium is given by*

$$U^B = (U_1^B, U_2^B) \div (-D_{21}^{-1}\Theta_2^B(t,\varepsilon)x, -\varepsilon D_{12}^{-1}\Theta_1^B(t,\varepsilon)x); \tag{5.11}$$

b) *the Berge equilibrium payoffs of the players are*

$$J_i(U^B, t_0, x_0) = x_0'\Theta_i^B(t_0,\varepsilon)x_0 \ \ (i = 1, 2). \tag{5.12}$$

P r o o f. Following Remark 4.3, we construct the two scalar functions (5.5).

Step 1. In view of (4.5) and $V_i^B(t,x) = x'\Theta_i^B(t)x$,

$$\Theta_i^B(\vartheta,\varepsilon) = C_i \ (i=1,2). \tag{5.13}$$

Step 2. Due to (4.6), using (1.3)–(1.5) we write

$$\begin{aligned}
&\left.\frac{\partial W_1(t,x,u_1^B(t,x,V^B),u_2,V^B)}{\partial u_2}\right|_{u_2(t,x,V^B)} = \varepsilon\frac{\partial V_1}{\partial x} + 2D_{12}u_2^B(t,x,V^B) = 0_n,\\
&\left.\frac{\partial^2 W_1(t,x,u_1^B(t,x,V^B),u_2,V^B)}{\partial u_2^2}\right|_{u_2(t,x,V^B)} = 2D_{12} < 0 \ \mathit{and}\\
&\left.\frac{\partial W_2(t,x,u_1,u_2^B(t,x,V^B),V^B)}{\partial u_1}\right|_{u_1(t,x,V^B)} = \frac{\partial V_2}{\partial x} + 2D_{21}u_1^B(t,x,V^B) = 0_n,\\
&\left.\frac{\partial^2 W_2(t,x,u_1,u_2^B(t,x,V^B),V^B)}{\partial u_1^2}\right|_{u_1(t,x,V^B)} = 2D_{21} < 0,
\end{aligned}$$

for any$(t,x) \in [0,\vartheta]\times[\mathbb{R}^n\backslash\{0_n\}]$ and $V^B = (V_1^B, V_2^B) \in \mathbb{R}^2$.

The second and fourth relations are true by (5.9). Using the first and third relations, we find

$$u_1^B(t,x,V^B) = -\frac{1}{2}D_{21}^{-1}\frac{\partial V_2^B}{\partial x}, \quad u_2^B(t,x,V^B) = -\frac{\varepsilon}{2}D_{12}^{-1}\frac{\partial V_1^B}{\partial x}. \tag{5.14}$$

Step 3. Substituting (5.14) into (4.7), we obtain the system of two partial differential equations with boundary conditions (4.5):

$$\begin{aligned}
0 &= W_1^B[t,x,V^B] = W_1(t,x,u_1^B(t,x,V^B),u_2^B(t,x,V^B),V^B) =\\
&= \frac{\partial V_1^B}{\partial t} + \left[\frac{\partial V_1^B}{\partial x}\right]'\left[A(t)x - \frac{1}{2}D_{21}^{-1}\frac{\partial V_2^B}{\partial x} - \frac{\varepsilon^2}{2}D_{12}^{-1}\frac{\partial V_1^B}{\partial x}\right] +\\
&+ \frac{1}{4}\left[\frac{\partial V_2^B}{\partial x}\right]' D_{21}^{-1}D_{11}D_{21}^{-1}\frac{\partial V_2^B}{\partial x} + \frac{\varepsilon^2}{4}\left[\frac{\partial V_1^B}{\partial x}\right]' D_{12}^{-1}\frac{\partial V_1^B}{\partial x} =\\
&= \frac{\partial V_1^B}{\partial t} + \left[\frac{\partial V_1^B}{\partial x}\right]' A(t)x + \frac{1}{4}\left[\frac{\partial V_2^B}{\partial x}\right]' D_{21}^{-1}D_{11}D_{21}^{-1}\frac{\partial V_2^B}{\partial x} -\\
&- \frac{1}{2}\left[\frac{\partial V_1^B}{\partial x}\right]' D_{21}^{-1}\frac{\partial V_2^B}{\partial x} - \frac{\varepsilon^2}{4}\left[\frac{\partial V_1^B}{\partial x}\right]' D_{12}^{-1}\frac{\partial V_1^B}{\partial x},\\
0 &= W_2^B[t,x,V^B] = W_2(t,x,u_1^B(t,x,V^B),u_2^B(t,x,V^B),V^B) =\\
&= \frac{\partial V_2^B}{\partial t} + \left[\frac{\partial V_2^B}{\partial x}\right]'\left[A(t)x - \frac{1}{2}D_{21}^{-1}\frac{\partial V_2^B}{\partial x} - \frac{\varepsilon^2}{2}D_{12}^{-1}\frac{\partial V_1^B}{\partial x}\right] +\\
&+ \frac{1}{4}\left[\frac{\partial V_2^B}{\partial x}\right]' D_{21}^{-1}\frac{\partial V_2^B}{\partial x} + \frac{\varepsilon^2}{4}\left[\frac{\partial V_1^B}{\partial x}\right]' D_{12}^{-1}D_{22}D_{12}^{-1}\frac{\partial V_1^B}{\partial x} =\\
&= \frac{\partial V_2^B}{\partial t} + \left[\frac{\partial V_2^B}{\partial x}\right]' A(t)x - \frac{1}{4}\left[\frac{\partial V_2^B}{\partial x}\right]' D_{21}^{-1}\frac{\partial V_2^B}{\partial x} +\\
&+ \frac{\varepsilon^2}{4}\left[\frac{\partial V_1^B}{\partial x}\right]' D_{12}^{-1}D_{22}D_{12}^{-1}\frac{\partial V_1^B}{\partial x} - \frac{\varepsilon^2}{2}\left[\frac{\partial V_2^B}{\partial x}\right]' D_{12}^{-1}\frac{\partial V_1^B}{\partial x}.
\end{aligned}$$

Letting $V_i^B(t,x) = x'\Theta_i^B(t)x$ and $\frac{\partial V_i^B}{\partial x} = 2\Theta_i^B(t)x$, we demonstrate that due to (5.13) the previous equalities hold if $\Theta_i^B(t,\varepsilon)$ $(i = 1,2)$ are the solutions of the system (5.10). Using the resulting solution $(\Theta_1^B(t,\varepsilon), \Theta_2^B(t,\varepsilon))$ of the system (5.10), the explicit form of the functions $V_i^B(t,x) = x'\Theta_i^B x$ and gradients $\frac{\partial V_i^B}{\partial t} = 2\Theta_i^B x$ as well as the inclusions $D_{21}^{-1}\Theta_2^B(\cdot,\varepsilon)$, $\varepsilon D_{12}^{-1}\Theta_1^B(\cdot,\varepsilon) \in C^1_{n\times n}[0,\vartheta]$ we finally prove (5.14). Note that the relations (5.12) are true according to (4.8). □

Remark 5.2. By analogy with Remark 5.1, we can establish the following result: if $D_{12} > 0$ and/or $D_{21} > 0$, then there are no Berge equilibria in the game Γ_2 for any initial position $(t_0, x_0) \in [0,\vartheta) \times [\mathbb{R}^n \backslash \{0_n\}]$.

§ 6. Application of small parameter method

Poincaré theorem

Thus, Propositions 5.1 and 5.2 have showed that the presence of Berge and/or Nash equilibria is connected with the existence of a solution for the corresponding systems of two matrix ordinary differential equations of the Riccati type that can be extended to the entire interval $[0,\vartheta]$ of the game. As a matter of fact, the existence of solutions in a small left neighborhood $(\vartheta-\delta,\vartheta]$ of the point $t=\vartheta$ is guaranteed by general existence theorems from the theory of ordinary differential equations. The question of the extendability of such solutions to the entire interval $[0,\vartheta]$ of the game remains open. In this section, we will try to answer it using the small parameter method. This method arose in connection with the three-body problem in celestial mechanics; it dates back to J. D'Alembert and was intensively developed starting from the end of the 19 th century. Further, from the numerous theoretical results on the small parameter method [3, 4], we will use Poincaré's theorem on the analyticity of solutions with respect to the parameter. It will be formulated for the matrix system of ordinary differential equations

$$\dot{\Theta} = \Xi(t,\Theta,\varepsilon), \quad \Theta(\vartheta,\varepsilon) = C. \tag{6.1}$$

The notations are the following: Θ in a matrix of dimensions $n\times n$; $\Xi(t,\Theta,\varepsilon)$ in a matrix of dimensions $n\times n$ whose elements are functions of the variables t, Θ, and ε; ε in a small parameter such that $0 \leqslant \varepsilon \leqslant \varepsilon_0$, where ε_0 is a small number; C in a constant matrix of dimensions $n\times n$; $t\in[0,\vartheta]$ in continuous time. The elements of the matrix $\Xi(t,\Theta,\varepsilon)$ are assumed to be defined and continuous on domain G, $\varepsilon\in[0,\varepsilon_0]$. Denote by $\Theta = \Theta(t,\varepsilon)$ a solution of (6.1) that satisfies the boundary value conditions $\Theta(\vartheta,\varepsilon) = C$, $(\vartheta,\varepsilon)\in G$. Together with the system (6.1), consider the system

$$\dot{\Theta} = \Xi(t,\Theta,0), \quad \Theta(\vartheta,0) = C, \tag{6.2}$$

which is obtained from (6.1) for $\varepsilon = 0$. Let $\Theta = \Theta^{(0)}(t)$ be a solution of (6.2) defined on $t\in[0,\vartheta]$ with the same boundary-value condition $\Theta(\vartheta) = C$. For a small value ε, the right-hand sides of these systems are close to each other.Then a natural question is: how do the solutions of the systems (6.1) and (6.2) differ on the entire interval $[0,\vartheta]$? By the theorem on the continuous dependence of solutions of combined ordinary differential equations on the parameter, generally these solutions are close to each other too. Moreover, if there exists a unique solution $\Theta^{(0)}(t)$ of the system (6.2) and the elements of $\Xi(t,\Theta,\varepsilon)$ are holomorphic (analytic) for $0\leqslant\varepsilon\leqslant\varepsilon_0$, $\Theta = \Theta^{(0)}(t)$, $t\in[0,\vartheta]$, then for a sufficiently small value ε the solution of (6.1) can be written as the series

$$\Theta(t,\varepsilon) = \Theta^{(0)}(t) + \sum_{m=1}^{\infty}\varepsilon^m\Theta^{(m)}(t),$$

which has uniform convergence on the entire interval $[0,\vartheta]$. This fact is the core of *Poincaré's theorem*.

Among general theorems from the theory of differential equations, also we will employ the theorem on the continuous dependence of solutions on the parameter; see below.

Theorem 6.1. *Let the right-hand side of system* (6.1) *be continuously differentiable with respect to the elements of the matrix* Θ *and also continuous in* ε *on the domain* G. *Then for a sufficiently small value* $\varepsilon > 0$, *the solution* $\Theta(t,\varepsilon)$ *of system* (6.1) *is well-defined on the same interval* $[0,\vartheta]$ *as the solution of system* (6.2).

Nash equlubrium

We will demonstrate that the existence of a solution $(\Theta_1^e(t,\varepsilon), \Theta_2^e(t,\varepsilon))$ of the system (5.2) that is extendable to $[0,\vartheta]$ is a superfluous requirement of Proposition 5.1 in the case of a small value $\varepsilon > 0$. In other words, it can be neglected for sufficiently small values $\varepsilon > 0$. More specifically, we will establish the following result.

Proposition 6.1. *Consider the game* Γ_2 *with the matrices*

$$D_{11} < 0, \quad D_{22} < 0, \quad C_1 < 0.$$

Then for sufficiently small values $\varepsilon > 0$ *the game* Γ_2 *has the Nash equilibrium* (5.3) *and the corresponding payoffs of the players are given by* (5.4).

P r o o f. Proposition 6.1 can be proved by demonstrating that the system (5.2) with sufficiently small values $\varepsilon > 0$ has a solution $(\Theta_1^e(t,\varepsilon), \Theta_2^e(t,\varepsilon))$, $t \in [0,\vartheta]$, that is extendable to $[0,\vartheta]$.

To this end, we will utilize Theorem 6.1. For (5.2), we construct the null approximation by letting $\varepsilon = 0$. In this case, the system (5.2) is decomposed into two subsystems of matrix ordinary differential equations. One of them belongs to the Riccati class whereas the other is linear in $\Theta_2^{(0)}$:

$$\begin{cases} \dot\Theta_1^{(0)} + \Theta_1^{(0)} A(t) + A'(t)\Theta_1^{(0)} - \Theta_1^{(0)} D_{11}^{-1} \Theta_1^{(0)} = O_{n\times n}, \quad \Theta_1^{(0)}(\vartheta) = C_1, \\ \dot\Theta_2^{(0)} + \Theta_2^{(0)} \left[A(t) - D_{11}^{-1}\Theta_1^{(0)}\right] + \left[A'(t) - \Theta_1^{(0)} D_{11}^{-1}\right] \Theta_2^{(0)} + \\ + \Theta_1^{(0)} D_{11}^{-1} D_{12} D_{11}^{-1} \Theta_1^{(0)} = O_{n\times n}, \quad \Theta_2^{(0)}(\vartheta) = C_2. \end{cases} \tag{6.3}$$

For $D_{11} < 0$, $C_1 < 0$ and $A(\cdot) \in C^{n\times n}[0,\vartheta]$ the solution $\Theta_1^{(0)}(t)$ of the first part of the system (6.3) exists, is continuous and extendable to $[0,\vartheta]$, symmetric $([\Theta_1^{(0)}(t)]' = \Theta_1^{(0)}(t))$ and negative $(\Theta_1^{(0)}(t) < 0)$ for all $t \in [0,\vartheta]$ and has the form

$$\Theta_1^{(0)}(t) = [X^{-1}(t)]' \left\{ C_1^{-1} + \int_t^{\vartheta} X^{-1}(\tau) D_{11}^{-1} [X^{-1}(\tau)]' \, d\tau \right\}^{-1} X^{-1}(t), \tag{6.4}$$

where $X(t)$ denotes the fundamental matrix for $\dot x = A(t)x$, $X(\vartheta) = E_n$; see Proposition 2.1. Incorporating this matrix $\Theta_1^{(0)} = \Theta_1^{(0)}(t)$ into the second part of the system, we obtain the following matrix linear inhomogeneous differential equation in $\Theta_2^{(0)}$:

$$\begin{aligned} &\dot\Theta_2^{(0)} + \Theta_2^{(0)} \left[A(t) - D_{11}^{-1}\Theta_1^{(0)}(t)\right] + \left[A'(t) - \Theta_1^{(0)}(t) D_{11}^{-1}\right] \Theta_2^{(0)} + \\ &+ \Theta_1^{(0)}(t) D_{11}^{-1} D_{12} D_{11}^{-1} \Theta_1^{(0)}(t) = O_{n\times n}, \quad \Theta_1^{(0)}(\vartheta) = C_2. \end{aligned} \tag{6.5}$$

Since $\Theta_1^{(0)}(\cdot) \in C^1_{n\times n}[0,\vartheta]$, $A(\cdot) \in C^{n\times n}[0,\vartheta]$, for any constant matrix C_2 of dimensions $n \times n$ equation (6.5) has a continuous and extendable to $[0,\vartheta]$ solution of the form

$$\Theta_2^{(0)}(t) = [X^{-1}(t)]' \left\{ C_2 + \int_t^{\vartheta} X'(\tau) B_1(\tau) X(\tau)\, d\tau \right\} X^{-1}(t), \tag{6.6}$$

with the continuous and symmetric matrix

$$B_1(t) = \Theta_1^{(0)}(t) D_{11}^{-1} D_{12} D_{11}^{-1} \Theta_1^{(0)}(t).$$

of dimensions $n \times n$; see Proposition 2.2. From (6.6) and the symmetry of C_2 and $B(t)$, it follows that (6.6) holds for any $t \in [0,\vartheta]$ (like in (6.4), $X(t)$ denotes the fundamental matrix). Consequently, the system (5.2) with $\varepsilon = 0$ has a continuous and extendable to $[0,\vartheta]$ solution $(\Theta_1^{(0)}(t), \Theta_2^{(0)}(t))$. Therefore, by Theorem 6.1 the system (5.2) with sufficiently small values $\varepsilon > 0$ also has an extendable to $[0,\vartheta]$ solution $(\Theta_1^e(t,\varepsilon), \Theta_2^e(t,\varepsilon))$. And Proposition 6.1 directly follows from Proposition 5.1. □

Berge equilibrium

Like for Nash equilibrium, we will demonstrate that the existence of a solution $(\Theta_1^B(t,\varepsilon),$ $\Theta_2^B(t,\varepsilon))$ of the system (5.10) that is extendable to $[0,\vartheta]$ is a superfluous requirement of Proposition 5.2, which can be replaced by the smallness of $\varepsilon > 0$.

Proposition 6.2. *Consider the game Γ_2 with the matrices*

$$D_{12} < 0, \quad D_{21} < 0, \quad C_2 < 0.$$

Then for sufficient small values $\varepsilon > 0$, the game Γ_2 has the Berge equilibrium (5.11) and the corresponding payoffs of the players are given by (5.12).

P r o o f. As before, using Theorem 6.1 we will prove that the solution of the system (5.10) is extendable to $[0,\vartheta]$. By analogy with Proposition 6.1 we construct the null approximation $(\widetilde{\Theta}_1^{(0)}(t), \widetilde{\Theta}_2^{(0)}(t))$ by letting $\varepsilon = 0$ in (5.10). As a result, the system (5.10) is decomposed into the two subsystems of matrix nonlinear differential equations

$$\begin{cases} \dot{\widetilde{\Theta}}_1^{(0)} + \widetilde{\Theta}_1^{(0)} \left[A(t) - D_{21}^{-1} \widetilde{\Theta}_2^{(0)} \right] + \left[A'(t) - \widetilde{\Theta}_2^{(0)} D_{21}^{-1} \right] \widetilde{\Theta}_1^{(0)} + \\ + \widetilde{\Theta}_2^{(0)} D_{21}^{-1} D_{11} D_{21}^{-1} \widetilde{\Theta}_2^{(0)} = O_{n\times n}, \quad \widetilde{\Theta}_1^{(0)}(\vartheta) = C_1, \\ \dot{\widetilde{\Theta}}_2^{(0)} + \widetilde{\Theta}_2^{(0)} A(t) + A'(t) \widetilde{\Theta}_2^{(0)} - \widetilde{\Theta}_2^{(0)} D_{21}^{-1} \widetilde{\Theta}_2^{(0)} = O_{n\times n}, \quad \widetilde{\Theta}_2^{(0)}(\vartheta) = C_2. \end{cases} \tag{6.7}$$

For $D_{21} < 0$ and $C_2 < 0$, the solution $\widetilde{\Theta}_2^{(0)}(t)$ for the matrix system of Riccati differential equations (the second equation in (6.7)) exists, is continuous and extendable to $[0,\vartheta]$, symmetric $([\widetilde{\Theta}_2^{(0)}(t)]' = \widetilde{\Theta}_2^{(0)}(t))$ and negative $(\widetilde{\Theta}_2^{(0)}(t) < 0)$ for all $t \in [0,\vartheta]$ and has the form

$$\widetilde{\Theta}_2^{(0)}(t) = [X^{-1}(t)]' \left\{ C_2^{-1} + \int_t^{\vartheta} X^{-1}(\tau) D_{21}^{-1} [X^{-1}(\tau)]'\, d\tau \right\}^{-1} X^{-1}(t). \tag{6.8}$$

Incorporating the solution $\widetilde{\Theta}_2^{(0)} = \widetilde{\Theta}_2^{(0)}(t)$ into the first part of (6.7) we obtain the following matrix linear inhomogeneous ordinary differential equation in $\widetilde{\Theta}_1^{(0)}$:

$$\dot{\widetilde{\Theta}}_1^{(0)} + \widetilde{\Theta}_1^{(0)} \left[A(t) - D_{21}^{-1} \widetilde{\Theta}_2^{(0)}(t) \right] + \left[A'(t) - \widetilde{\Theta}_2^{(0)}(t) D_{21}^{-1} \right] \widetilde{\Theta}_1^{(0)} +$$
$$+ \widetilde{\Theta}_2^{(0)}(t) D_{21}^{-1} D_{11} D_{21}^{-1} \widetilde{\Theta}_2^{(0)}(t) = O_{n\times n}, \quad \widetilde{\Theta}_1^{(0)}(\vartheta) = C_1.$$

In view of the inclusions $\widetilde{\Theta}_2^{(0)}(\cdot) \in C^1_{n\times n}[0,\vartheta]$, $A(\cdot) \in C^{n\times n}[0,\vartheta]$ and Proposition 2.2, the explicit solution is given by

$$\widetilde{\Theta}_1^{(0)}(t) = [X^{-1}(t)]' \left\{ C_1 + \int_t^{\vartheta} X'(\tau)B_2(\tau)X(\tau)\, d\tau \right\} X^{-1}(t), \tag{6.9}$$

with the continuous and symmetric matrix

$$B_2(t) = \widetilde{\Theta}_2^{(0)}(t)D_{21}^{-1}D_{11}D_{21}^{-1}\widetilde{\Theta}_2^{(0)}(t)$$

of dimensions $n \times n$.

Clearly, the continuous matrix $\widetilde{\Theta}_1^{(0)}(t)$ of dimensions $n \times n$ is well-defined for all $t \in [0,\vartheta]$ and symmetric. Hence, for $\varepsilon = 0$ the null approximation $(\widetilde{\Theta}_1^{(0)}(t), \widetilde{\Theta}_2^{(0)}(t) \big| t \in [0,\vartheta])$ of the solution $(\Theta_1^B(t,\varepsilon), \Theta_2^B(t,\varepsilon) \big| t \in [0,\vartheta])$ of the system (5.10) is extendable to $[0,\vartheta]$. By Theorem 6.1 the system (5.10) with sufficiently small values $\varepsilon > 0$ has an extendable to $[0,\vartheta]$ solution $(\Theta_1^B(t,\varepsilon), \Theta_2^B(t,\varepsilon))$. And Proposition 6.2 directly follows from Proposition 5.2. $\square$

§ 7. Coefficient criteria of existence

This section is devoted to the coefficient criteria of the existence (and nonexistence!) of Nash and/or Berge equilibria (in terms of Definitions 4.1 and 4.2 respectively) in the differential positional linear-quadratic game Γ_2 with a small influence of one player on the rate of change $\dot{x}(t)$ of the state vector $x(t)$. In the game Γ_2 the state vector evolves in accordance with the vector linear differential equation

$$\dot{x} = A(t)x + u_1 + \varepsilon u_2, \quad x(t_0) = x_0,$$

and the payoff function of player i is described by the quadratic functional

$$J_i(U_1, U_2, t_0, x_0) = x'(\vartheta)C_i x(\vartheta) + \int_{t_0}^{\vartheta} \{u_1'[t]D_{i1}u_1[t] + u_2'[t]D_{i2}u_2[t]\}\, dt \quad (i = 1, 2),$$

where $x,\ u_i \in \mathbb{R}^n$. As before, the prime indicates transposition. The strategy set of player i has the form

$$\mathbf{U}_i = \left\{ U_i \div u_i(t,x) \mid u_i(t,x) = Q_i(t)x \ \forall Q(\cdot) \in C^{n\times n}[0,\vartheta] \right\};$$

the game ends at a fixed time instant $\vartheta > t_0 \geqslant 0$; the symmetric constant matrices C_i and D_{ij} of dimensions $n \times n$ are given; the notation $D > 0$ (< 0) means that a quadratic form $x'Dx$ is positive definite (negative definite, respectively); $\varepsilon \geqslant 0$ is a small scalar parameter. The players choose their strategies $U_i \div Q_i(t)x$, find the solution $x(t)$ of the system equation

$$\dot{x} = A(t)x + Q_1(t)x + \varepsilon Q_2(t)x, \quad x(t_0) = x_0,$$

construct the realizations $u_i[t] = Q_i(t)x(t)$ of the chosen strategies U_i and then calculate their payoffs $J_i(U_1, U_2, t_0, x_0)$ using $x(t)$ and $u_i[t]$.

In the noncooperative statement of the game Γ_2 the players have to answer two questions as follows.

1. Which of the solution concepts (Nash or Berge equilibrium) should they adhere to?

2. How can these equilibria be constructed?

The answer to the first question is provided by Table 1. Here NE and BE denote Nash and Berge equilibrium, respectively; $\exists$, $\nexists$ and $\forall$ are the existential, non-existential and universal quantifiers, respectively. Proposition 6.1 and 6.2 as well as Remarks 5.1 and 5.2 are combined in Table 1, which presents the coefficient criteria of choosing (or rejecting) Nash and/or Berge equilibrium in the game Γ_2.

For example, if $D_{12} < 0$, $D_{21} < 0$, $C_2 < 0$ then there exists a Berge equilibrium; if simultaneously $D_{22} > 0$, then there does not exist a Nash equilibrium (see columns 2 and 7 of the table below).

Table 1. Coefficient criteria of equilibrium

	D_{11}	D_{12}	D_{21}	D_{22}	C_1	C_2	NE	BE
1	$D_{11} < 0$	$\forall$	$\forall$	$D_{22} < 0$	$C_1 < 0$	$\forall$	$\exists$	
2	$\forall$	$D_{12} < 0$	$D_{21} < 0$	$\forall$	$\forall$	$C_2 < 0$		$\exists$
3	$D_{11} < 0$	$D_{12} < 0$	$D_{21} < 0$	$D_{22} < 0$	$C_1 < 0$	$C_2 < 0$	$\exists$	$\exists$
4	$D_{11} > 0$	$\forall$	$\forall$	$\forall$	$\forall$	$\forall$	$\nexists$	
5	$\forall$	$D_{12} > 0$	$\forall$	$\forall$	$\forall$	$\forall$		$\nexists$
6	$\forall$	$\forall$	$D_{21} > 0$	$\forall$	$\forall$	$\forall$		$\nexists$
7	$\forall$	$\forall$	$\forall$	$D_{22} > 0$	$\forall$	$\forall$	$\nexists$	

The answer to the second question is based on Poincaré's theorem; see the beginning of Section 6. More specifically, we have to consider not only the null term ($\varepsilon = 0$) of the matrix expansion

$$\Theta(t, \varepsilon) = \Theta^{(0)}(t) + \sum_{m=1}^{\infty} \varepsilon^m \Theta^{(m)}(t),$$

but also the subsequent ones $\Theta^{(1)}(t)$, $\Theta^{(2)}(t), \ldots$. This approach will be illustrated by an example of Berge equilibrium design for the game Γ_2: we will find the solution of (5.10) and then construct the strategies (5.11) and the corresponding payoffs (5.12). In view of $\Theta^B(t, \varepsilon) = \Theta_1^{(0)}(t) + \varepsilon^1 \Theta_1^{(1)}(t) + \varepsilon^2 \Theta_1^{(2)}(t) + \ldots$ and (5.10), we have

$$\begin{aligned}
&(\dot{\Theta}_1^{(0)} + \varepsilon^1 \dot{\Theta}_1^{(1)} + \varepsilon^2 \dot{\Theta}_1^{(2)} + \ldots) + (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots)[A(t) - \\
&- D_{21}^{-1}(\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots)] + \\
&+ \left[A'(t) - (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) D_{21}^{-1}\right] (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots) + \\
&+ (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) D_{21}^{-1} D_{11} D_{21}^{-1} (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) - \\
&- \varepsilon^2 (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots) D_{21}^{-1} (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots) = O_{n\times n}, \\
&(\Theta_1^{(0)}(\vartheta) + \varepsilon^1 \Theta_1^{(1)}(\vartheta) + \varepsilon^2 \Theta_1^{(2)}(\vartheta) + \ldots) = C_1,
\end{aligned} \tag{7.1}$$

$$\begin{aligned}
&(\dot{\Theta}_2^{(0)} + \varepsilon^1 \dot{\Theta}_2^{(1)} + \varepsilon^2 \dot{\Theta}_2^{(2)} + \ldots) + (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) A(t) + \\
&+ A'(t)(\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) - \\
&- (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) D_{21} (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) + \\
&+ \varepsilon^2[-(\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) D_{12}^{-1} (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots) - \\
&- (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots) D_{12}^{-1} (\Theta_2^{(0)} + \varepsilon^1 \Theta_2^{(1)} + \varepsilon^2 \Theta_2^{(2)} + \ldots) + \\
&+ (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots) D_{12}^{-1} D_{22} D_{12}^{-1} (\Theta_1^{(0)} + \varepsilon^1 \Theta_1^{(1)} + \varepsilon^2 \Theta_1^{(2)} + \ldots)] = O_{n\times n}, \\
&(\Theta_2^{(0)}(\vartheta) + \varepsilon^1 \Theta_2^{(1)}(\vartheta) + \varepsilon^2 \Theta_2^{(2)}(\vartheta) + \ldots) = C_2.
\end{aligned} \tag{7.2}$$

According to the proof of Proposition 6.2, the null approximations $\Theta_i^{(0)}(t)$ $(i=1,2)$ satisfy the system (6.7) and have the explicit forms (6.8) and (6.9) respectively, where $\widetilde{\Theta}_i^{(0)}(t)=\Theta_i^{(0)}(t)$ $\forall t\in[0,\vartheta]$, $(i=1,2)$. Equalizing the terms with the factor ε in (7.1)–(7.2), we obtain the following system of two matrix linear homogeneous differential equations with time-continuous coefficients to calculate the first approximations:

$$\begin{cases} \dot{\Theta}_1^{(1)}+\Theta_1^{(1)}\left[A(t)-D_{21}^{-1}\Theta_2^{(0)}(t)\right]+\left[A'(t)-\Theta_2^{(0)}(t)D_{21}^{-1}\right]\Theta_1^{(1)}+\\ +\Theta_2^{(1)}D_{21}^{-1}[D_{11}D_{21}^{-1}\Theta_2^{(0)}(t)-\Theta_1^{(0)}(t)]+[\Theta_2^{(0)}(t)D_{21}^{-1}D_{11}-\Theta_1^{(0)}(t)]D_{21}^{-1}\Theta_2^{(1)}=O_{n\times n},\\ \Theta_1^{(1)}(\vartheta)=O_{n\times n},\\ \dot{\Theta}_2^{(1)}+\Theta_2^{(1)}\left[A(t)-D_{21}^{-1}\Theta_2^{(0)}(t)\right]+\left[A'(t)-\Theta_2^{(0)}(t)D_{21}^{-1}\right]\Theta_2^{(1)}=O_{n\times n},\\ \Theta_2^{(1)}(\vartheta)=O_{n\times n}. \end{cases}$$

Obviously, it has the trivial solution

$$\Theta_1^{(1)}(t)=\Theta_2^{(1)}(t)=O_{n\times n}\quad\forall t\in[0,\vartheta]. \tag{7.3}$$

Now, equalizing the terms with the factor ε^2 in (7.1)–(7.2) and using (7.3), we derive the following system of two matrix linear inhomogeneous differential equations with time-continuous coefficients to calculate the second approximations $\Theta_1^{(2)}(t)$ and $\Theta_2^{(2)}(t)$:

$$\begin{aligned} &\dot{\Theta}_1^{(2)}+\Theta_1^{(2)}\left[A(t)-D_{21}^{-1}\Theta_2^{(0)}(t)\right]+\left[A'(t)-\Theta_2^{(0)}(t)D_{21}^{-1}\right]\Theta_1^{(2)}+\\ &+\Theta_2^{(2)}D_{21}^{-1}[D_{11}D_{21}^{-1}\Theta_2^{(0)}(t)-\Theta_1^{(0)}(t)]+[\Theta_2^{(0)}(t)D_{21}^{-1}D_{11}-\Theta_1^{(0)}(t)]D_{21}^{-1}\Theta_2^{(2)}-\\ &-\Theta_1^{(0)}(t)D_{12}^{-1}\Theta_1^{(0)}(t)=O_{n\times n},\quad \Theta_1^{(2)}(\vartheta)=O_{n\times n}, \end{aligned} \tag{7.4}$$

$$\begin{aligned} &\dot{\Theta}_2^{(2)}+\Theta_2^{(2)}\left[A(t)-D_{21}^{-1}\Theta_2^{(0)}(t)\right]+\left[A'(t)-\Theta_2^{(0)}(t)D_{21}^{-1}\right]\Theta_1^{(2)}+\\ &+\Theta_1^{(0)}D_{12}^{-1}D_{22}D_{12}^{-1}\Theta_1^{(0)}(t)-\Theta_2^{(0)}(t)D_{12}^{-1}\Theta_1^{(0)}(t)-\Theta_1^{(0)}(t)D_{12}^{-1}\Theta_2^{(0)}(t)=O_{n\times n},\\ &\Theta_2^{(2)}(\vartheta)=O_{n\times n}. \end{aligned} \tag{7.5}$$

We find the explicit-form solution of (7.4)–(7.5). First, using Proposition 2.2 we construct the solution $\Theta_2^{(2)}(t)$ of the second matrix equation from (7.4)–(7.5). For this purpose, we write the fundamental matrix $Y(t)$ for the vector differential equation $(y\in\mathbb{R}^n)$:

$$\dot{y}=[A(t)-D_{21}^{-1}\Theta_2^{(0)}(t)]y,\quad Y(\vartheta)=E_n.$$

According to Proposition 2.2 the solution of (7.5) takes the form

$$\Theta_2^{(2)}(t)=[Y^{-1}(t)]'\left\{\int_t^{\vartheta}Y'(\tau)L(\tau)Y(\tau)\,d\tau\right\}Y^{-1}(t),$$

where

$$L(t)=\Theta_1^{(0)}D_{12}^{-1}D_{22}D_{12}^{-1}\Theta_1^{(0)}(t)-\Theta_2^{(0)}(t)D_{12}^{-1}\Theta_1^{(0)}(t)-\Theta_1^{(0)}(t)D_{12}^{-1}\Theta_2^{(0)}(t).$$

Substituting $\Theta_2^{(2)}=\Theta_2^{(2)}(t)$ into (7.4) we obtain a matrix linear inhomogeneous differential equation with the null boundary-value condition. Its explicit solution $\Theta_1^{(2)}(t)$, like the solution of the second equation from (7.4) is found using Proposition 2.2. Finally, with the resulting approximations $\Theta_i^{(j)}(t)$ $(j = 0, 1, 2;\ i = 1, 2)$, (3.3) and (3.4) the Berge equilibrium in the game Γ_2 can be written as

$$U^B = (U_1^B, U_2^B) \div (-D_{21}^{-1}[\Theta_2^{(0)}(t) + \varepsilon^2\Theta_2^{(2)}(t)]x, -\varepsilon D_{12}[\Theta_1^{(0)}(t) + \varepsilon^2\Theta_1^{(2)}(t)]x).$$

(The accuracy is up to the second approximation.) The corresponding payoffs of the players are given by

$$J_1(U^B, t_0, x_0) = x_0'[\Theta_1^{(0)}(t_0) + \varepsilon^2\Theta_1^{(2)}(t_0)]x_0,$$

$$J_2(U^B, t_0, x_0) = x_0'[\Theta_2^{(0)}(t_0) + \varepsilon^2\Theta_2^{(2)}(t_0)]x_0.$$

Concluding this paper, we suggest that the solution of any game (in particular, Γ_2) should be described by a pair

$$\left(U^S = (U_1^S, U_2^S),\ J^S = (J_1(U^S, t_0, x_0), J_2(U^S, t_0, x_0))\right).$$

In this case, a strategy profile U^S determines the behavioral rules of the players, and J^S their payoffs gained.

Finally, we point out that the approaches set out in this paper can be applied to investigations of multicriteria problems [13], non-cooperative (Nash and Berge equilibrium, equilibrium of objections and counter-objections) [10, 11, 15], cooperative (the Shapley value and C-core, etc.) [6–8], hierarchical (Stackelberg equilibrium, Hermeyer equilibrium) [1, 2, 5] and coalitional (coalitional equiulibrium) [6, 12] positional differential games.

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В. И. Жуковский, Л. В. Жуковская, С. Н. Сачков, Е. Н. Сачкова

Применение метода малого параметра Ляпунова–Пуанкаре для построения равновесия по Нэшу и Бержу в одной дифференциальной игре двух лиц

Ключевые слова: метод малого параметра, дифференциальная линейно-квадратичная бескоалиционная игра, равновесие по Нэшу, равновесие по Бержу.

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Метод малого параметра Пуанкаре активно применяется в небесной механике, а также в теории дифференциальных уравнений и в ее важном разделе — оптимальном управлении. В предлагаемой статье данный метод используется для построения явного вида равновесия по Нэшу и Бержу в дифференциальной позиционной игре с малым влиянием одного из игроков на скорость изменения фазового вектора.

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