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ON THE GROWTH OF SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS IN $\overline{\mathbb{C}}\backslash\{z_0\}$ OF FINITE LOGARITHMIC ORDER

In this article, we study the growth of solutions of homogeneous and non-homogeneous complex linear differential equations where the coefficients are analytic functions in the extended complex plane except a finite singular point with finite logarithmic order. We extend some previous results obtained very recently by Fettouch and Hamouda.

Keywords: linear differential equation, analytic function, singular point, logarithmic order, logarithmic type.

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§ 1. Introduction and Main Results

Throughout this paper, we assume the reader is familiar with the fundamental results and standard notations of the Nevanlinna distribution theory of meromorphic functions (see [16, 17, 22]). Recently, there were interesting results on the growth of solutions of the complex linear differential equations by using a new idea in which the coefficients are analytic functions in the extended complex plane except a finite singular point $\overline{\mathbb{C}}\backslash\{z_0\}$ with non-zero positive order (see e. g. [6, 7, 10, 12, 13, 15, 18, 19]), which are similar to some of those obtained for the case when the coefficients are non-zero order entire functions (see e. g. [1,14,20,21]). The concept of logarithmic order due to Chern [8, 9] was used to investigate the growth of solutions to linear differential equations, difference and differential-difference equations for the case when the coefficients are zero order entire or meromorphic functions (see e.g. [2–5,11]). In this article, we continue making use of this concept to investigate the growth of solutions to homogeneous and non-homogeneous linear differential equations in which the coefficients are analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ with zero order, where we generalize those results obtained in [13]. We start by stating some essential definitions: for all $R \in (0,\infty)$ and $p \geq 1$, we define $\exp_1 R = e^R$, $\exp_{p+1} R = \exp(\exp_p R)$, $\log_1 R = \log R$ and $\log_{p+1} R = \log(\log_p R)$.

Definition 1.1 ([12]). Let f be a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, where $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, $z_0 \in \mathbb{C}$. The counting function of f near z_0 is defined by

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f)-n(\infty,f)}{t}dt - n(\infty,f)\log r,$$

where $n(t,f)$ counts the number of poles of f in $\{z \in \mathbb{C}\colon t \leq |z-z_0|\} \cup \{\infty\}$, each pole according to its multiplicity. The proximity function of f near z_0 is defined by

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})|\, d\phi.$$

The characteristic function of f near z_0 is defined by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

Definition 1.2 ([19]). Let f be a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, p and q be two integers with $p \geq q \geq 1$. The $[p,q]$-order of f near z_0 is defined by

$$\sigma_{[p,q]}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ T_{z_0}(r, f)}{\log_q \frac{1}{r}}.$$

For an analytic function f in $\overline{\mathbb{C}}\backslash\{z_0\}$, the $[p,q]$-order of f near z_0 is defined by

$$\sigma_{[p,q]}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ T_{z_0}(r, f)}{\log_q \frac{1}{r}} = \limsup_{r\longrightarrow 0} \frac{\log_{p+1}^+ M_{z_0}(r, f)}{\log_q \frac{1}{r}},$$

where $M_{z_0}(r,f) = \max\{|f(z)|\colon |z - z_0| = r\}$. If $\sigma_{[p,q]}(f, z_0) = \sigma \in (0, \infty)$, then the $[p,q]$-type of a meromorphic function f in $\overline{\mathbb{C}}\backslash\{z_0\}$ is defined by

$$\tau_{[p,q]}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_{p-1}^+ T_{z_0}(r, f)}{(\log_{q-1} \frac{1}{r})^\sigma}.$$

For an analytic function f in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\sigma_{[p,q]}(f, z_0) = \sigma \in (0, \infty)$, the $[p,q]$-type of f near z_0 is defined by

$$\tau_{[p,q],M}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ M_{z_0}(r, f)}{(\log_{q-1} \frac{1}{r})^\sigma}.$$

The $[p,q]$ exponent of convergence of the sequence of $a-$points and distinct a-points of a meromorphic function f in $\overline{\mathbb{C}}\backslash\{z_0\}$ are respectively defined by

$$\lambda_{[p,q]}(f - a, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ N_{z_0}(r, \frac{1}{f-a})}{\log_q \frac{1}{r}}, \quad \overline{\lambda}_{[p,q]}(f - a, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ \overline{N}_{z_0}(r, \frac{1}{f-a})}{\log_q \frac{1}{r}}.$$

Remark 1.1. By Definition 1.2, we can see that $\sigma_{[p,1]}(f, z_0) = \sigma_p(f, z_0)$, $\tau_{[p,1]}(f, z_0) = \tau_p(f, z_0)$, $\lambda_{[p,1]}(f - a, z_0) = \lambda_p(f - a, z_0)$ and $\overline{\lambda}_{[p,1]}(f - a, z_0) = \overline{\lambda}_p(f - a, z_0)$ denote respectively the iterated p-order, the iterated p-type and the iterated p-exponent of convergence of a-points and distinct a-points (see, [13]).

By the original definitions of the logarithmic order and the logarithmic type [8, 9], we define the logarithmic order and logarithmic type of a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ as follows.

Definition 1.3. Let f be a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, the logarithmic order of f near z_0 is defined by

$$\sigma_{[1,2]}(f, z_0) = \sigma_{\log}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log^+ T_{z_0}(r, f)}{\log\log \frac{1}{r}}.$$

If f is an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, then

$$\sigma_{\log}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log^+ T_{z_0}(r, f)}{\log\log \frac{1}{r}} = \limsup_{r\longrightarrow 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{\log\log \frac{1}{r}}.$$

Definition 1.4. Let f be a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with logarithmic order $\sigma_{\log}(f, z_0) = \sigma \in [1, \infty)$, the logarithmic type of f near z_0 is defined by

$$\tau_{[1,2]}(f, z_0) = \tau_{\log}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{T_{z_0}(r, f)}{(\log \frac{1}{r})^\sigma}.$$

If f is an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with order $\sigma_{\log}(f, z_0) = \sigma \in [1, \infty)$, then the logarithmic type of f near z_0 is defined by

$$\tau_{\log,M}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log^+ M_{z_0}(r, f)}{(\log \frac{1}{r})^\sigma}.$$

Remark 1.2. According to [12, Lemma 2.2] if f is a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, then $g(\omega) = f(z_0 - \frac{1}{\omega})$ is meromorphic in $\mathbb{C}$ and they satisfy

$$T(R, g) = T_{z_0}\left(\frac{1}{R}, f\right).$$

Consequently all the properties of the logarithmic order for the meromorphic functions in $\mathbb{C}$ are hold, such as all the non-constant rational functions which are analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ are of logarithmic order equalling one, where there is no transcendental meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ of logarithmic order less than one, further, constant functions have zero logarithmic order and there are no meromorphic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of logarithmic order between zero and one (see, [8, 9]).

In [13], Fettouch and Hamouda considered the following complex homogeneous and non-homogeneous linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1.2}$$

where $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ and $F(z)$ are analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$. Their results were for the case when the coefficients are of finite iterated p-order, where they obtained the following theorems.

Theorem A ([13]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ such that for real constants $\alpha, \beta, \mu, \theta_1, \theta_2$ and a positive integer p with $0 \le \beta < \alpha$, $\mu > 0$, $\theta_1 < \theta_2$, $1 \le p < \infty$, the following inequalities hold:*

$$|A_0(z)| \ge \exp_p\left\{\frac{\alpha}{r^\mu}\right\},$$

$$|A_j(z)| \le \exp_p\left\{\frac{\beta}{r^\mu}\right\}, \qquad j = 1, \ldots, k-1,$$

where $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$ *and* $|z_0 - z| = r \to 0$. *Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of* (1.1) *satisfies $\sigma_{p+1}(f, z_0) \ge \mu$.*

Theorem B ([13]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ and $E \subset (0, 1)$ be a set of infinite logarithmic measure such that*

$$|A_0(z)| \ge \exp_p\left\{\frac{\alpha}{r^\mu}\right\},$$

$$|A_j(z)| \le \exp_p\left\{\frac{\beta}{r^\mu}\right\}, \qquad j = 1, \ldots, k-1,$$

with $0 \le \beta < \alpha$, $\mu > 0$ and $|z_0 - z| = r \to 0$, $r \in E$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.1) *satisfies $\sigma_{p+1}(f, z_0) \ge \mu$.*

Theorem C ([13]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite iterated order with* $\max\{\sigma_p(A_j, z_0) \colon j \ne 0\} \le \sigma_p(A_0, z_0) = \sigma < +\infty$, $1 < p < \infty$, *and $E \subset (0, 1)$ be a set of infinite logarithmic measure such that for some constants $0 \le \beta < \alpha$ and any given $\varepsilon > 0$, we have*

$$|A_0(z)| \ge \exp_p\left\{\frac{\alpha}{r^{\sigma-\varepsilon}}\right\}, \qquad |A_j(z)| \le \exp_p\left\{\frac{\beta}{r^{\sigma-\varepsilon}}\right\}, \quad j = 1, \ldots, k-1,$$

as $r \to 0$ with $r \in E$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.1) *satisfies $\sigma_{p+1}(f, z_0) = \sigma_p(A_0, z_0) = \sigma$.*

Theorem D ([13]). *Let* $A_0(z),\dots,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *satisfying*

$$\max\left\{\sigma_p(A_j,z_0)\colon j\neq 0\right\}<\sigma_p(A_0,z_0).$$

Then, every analytic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (1.1) *satisfies* $\sigma_{p+1}(f,z_0)=\sigma_p(A_0,z_0)$.

Theorem E ([13]). *Let* $A_0(z),\dots,A_{k-1}(z)$ *satisfy the hypotheses of Theorem C, and let* $F(z)\not\equiv 0$ *be an analytic function in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *with* $i(F)=q$.

i) If $q<p+1$ *or* $q=p+1$, $\sigma_{p+1}(F,z_0)<\sigma_p(A_0,z_0)$, *then every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (1.2) *satisfies* $\overline{\lambda}_{p+1}(f,z_0)=\lambda_{p+1}(f,z_0)=\sigma_{p+1}(f,z_0)=\sigma_p(A_0,z_0)$, *with at most one exceptional solution* f_0 *satisfying* $i(f_0)<p+1$ *or* $\sigma_{p+1}(f,z_0)<\sigma_p(A_0,z_0)$.

ii) If $q>p+1$ *or* $q=p+1$, $\sigma_p(A_0,z_0)<\sigma_{p+1}(F,z_0)<+\infty$, *then every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (1.2) *satisfies* $i(f)=q$ *and* $\sigma_q(f,z_0)=\sigma_q(F,z_0)$.

It is clear that the above results do not include the case when the coefficients are analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of order zero. Therefore, for that case here we use the logarithmic order in order to express the growth of solutions of the complex linear differential equations (1.1) and (1.2), where we obtain the following theorems.

Theorem 1.1. *Let* $A_0(z),\dots,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *such that, for real constants* $\alpha,\beta,\mu,\theta_1$ *and* θ_2 *with* $0\le\beta<\alpha$, $\mu\ge 1$, $\theta_1<\theta_2$,

$$|A_0(z)|\ge\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\mu}\right\},$$

$$|A_j(z)|\le\exp\left\{\beta\left(\log\frac{1}{r}\right)^{\mu}\right\},\qquad j=1,\dots,k-1,$$

where $\arg(z_0-z)=\theta\in(\theta_1,\theta_2)$ *and* $|z_0-z|=r\to 0$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (1.1) *satisfies* $\sigma_{[2,2]}(f,z_0)\ge\mu-1$.

Theorem 1.2. *Let* $A_0(z),\dots,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *and* $E\subset(0,1)$ *be a set of infinite logarithmic measure such that*

$$|A_0(z)|\ge\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\mu}\right\},$$

$$|A_j(z)|\le\exp\left\{\beta\left(\log\frac{1}{r}\right)^{\mu}\right\},\qquad j=1,\dots,k-1,$$

with $0\le\beta<\alpha$, $\mu\ge 1$ *and* $|z_0-z|=r\to 0$, $r\in E$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (1.1) *satisfies* $\sigma_{[2,2]}(f,z_0)\ge\mu-1$.

Theorem 1.3. *Let* $A_0(z),\dots,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\max\left\{\sigma_{\log}(A_j,z_0)\colon j\neq 0\right\}\le\sigma_{\log}(A_0,z_0)=\sigma<+\infty$ *and* $E\subset(0,1)$ *be a set of infinite logarithmic measure such that for some constants* $0\le\beta<\alpha$ *and any given* $\varepsilon>0$, *we have*

$$|A_0(z)|\ge\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\},$$

$$|A_j(z)|\le\exp\left\{\beta\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\},\qquad j=1,\dots,k-1,$$

as $r\to 0$ *with* $r\in E$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (1.1) *satisfies* $\sigma_{\log}(A_0,z_0)-1\le\sigma_{[2,2]}(f,z_0)\le\sigma_{\log}(A_0,z_0)=\sigma$.

Theorem 1.4. *Let $A_0(z),\dots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order with $\max\left\{\sigma_{\log}(A_j,z_0)\colon j\neq 0\right\} < \sigma_{\log}(A_0,z_0) = \sigma < +\infty$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.1) satisfies $\sigma_{\log}(A_0,z_0)-1\leq\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_0,z_0)$.*

Theorem 1.5. *Let $A_0(z),\dots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order with*

$$\max\left\{\sigma_{\log}(A_j,z_0)\colon j\neq 0\right\}\leq\sigma_{\log}(A_0,z_0)=\sigma,\quad 1\leq\sigma<+\infty,$$
$$\max\left\{\tau_{\log,M}(A_j,z_0)\colon \sigma_{\log}(A_j,z_0)=\sigma, j\neq 0\right\}<\tau_{\log,M}(A_0,z_0)=\tau<+\infty.$$

Then, every analytic solution $f(z)\not\equiv 0$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.1) satisfies $\sigma_{\log}(A_0,z_0)-1\leq$ $\leq\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_0,z_0)$.

Theorem 1.6. *Let $A_0(z),\dots,A_{k-1}(z)$ satisfy the hypotheses of Theorem 1.4 and let $F(z)(\not\equiv 0)$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$.*

i) If $\sigma_{\log}(A_0,z_0)\leq\sigma_{[2,2]}(F,z_0)<+\infty$, then every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.2) satisfies $\sigma_{[2,2]}(f,z_0)=\sigma_{[2,2]}(F,z_0)$.

ii) If $\sigma_{\log}(A_0,z_0)>\sigma_{[2,2]}(F,z_0)$, then every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.2) satisfies $\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_0,z_0)$, and that $\sigma_{[2,2]}(f,z_0)\geq\sigma_{\log}(A_0,z_0)-1$ with at most one exceptional solution, and that $\overline{\lambda}_{[2,2]}(f,z_0)=\lambda_{[2,2]}(f,z_0)=\sigma_{[2,2]}(f,z_0)$ holds for every solution f which satisfies $\sigma_{[2,2]}(f,z_0)=\sigma_{\log}(A_0,z_0)$.

Theorem 1.7. *Let $A_0(z),\dots,A_{k-1}(z)$ satisfy the hypotheses of Theorem 1.5 and let $F(z)(\not\equiv 0)$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$.*

i) If $\sigma_{\log}(A_0,z_0)\leq\sigma_{[2,2]}(F,z_0)<+\infty$, then every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.2) satisfies $\sigma_{[2,2]}(f,z_0)=\sigma_{[2,2]}(F,z_0)$.

ii) If $\sigma_{\log}(A_0,z_0)>\sigma_{[2,2]}(F,z_0)$, then every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.2) satisfies $\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_0,z_0)$, and that $\sigma_{[2,2]}(f,z_0)\geq\sigma_{\log}(A_0,z_0)-1$ with at most one exceptional solution, and that $\overline{\lambda}_{[2,2]}(f,z_0)=\lambda_{[2,2]}(f,z_0)=\sigma_{[2,2]}(f,z_0)$ holds for every solution f which satisfies $\sigma_{[2,2]}(f,z_0)=\sigma_{\log}(A_0,z_0)$.

Remark 1.3. We should point that similar results to those in Theorems 1.4–1.7 are obtained in [5] for the complex plane $\mathbb{C}$ case.

§ 2. Some preliminary lemmas

The following lemmas are important for proving our results. Firstly we denote the logarithmic measure of a set $E\subset(0,1)$ by $m_l(E)=\int_E\frac{dt}{t}$.

Lemma 2.1 ([12]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$, let $\lambda>0$, $\varepsilon>0$ be given real constants and $j\in\mathbb{N}$. Then:*

(i) there exist a set $E_1\subset(0,1)$ of finite logarithmic measure and a constant $C>0$ that depends only on λ and j such that for all $|z-z_0|=r\in(0,1)\backslash E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq C\left[\frac{1}{r^2}T_{z_0}(\lambda r,f)\log T_{z_0}(\lambda r,f)\right]^j; \tag{2.1}$$

(ii) there exist a set $E_2\subset[0,2\pi)$ that has a linear measure zero and a constant $C>0$ that depends on λ and j such that for all $\theta\in[0,2\pi)\backslash E_2$, there exists a constant $r_0=r_0(\theta)>0$ such that (2.1) holds for all z satisfying $\arg(z-z_0)\in[0,2\pi)\backslash E_2$ and $r=|z-z_0|<r_0$.

Lemma 2.2 ([15]). *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then there exists a set E_3 of $(0,1)$ that has finite logarithmic measure such that for all $j = 0, \ldots, k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j \big(1 + o(1)\big)$$

as $r \to 0$, $r \notin E_3$, where z_r is a point in the circle $|z - z_0| = r$ that satisfies $|f(z_r)| = \max_{|z-z_0|=r} |f(z)|$.

Lemma 2.3 ([19]). *Let p and q be two integers with $p \geq q \geq 1$. Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let $V_{z_0}(r)$ be the central index of f. Then*

$$\sigma_{[p,q]}(f, z_0) = \lim_{r \longrightarrow 0} \frac{\log_p^+ V_{z_0}(r)}{\log_q \frac{1}{r}}.$$

Lemma 2.4. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order with $\max\big\{\sigma_{\log}(A_j, z_0)\colon j = 0, \ldots, k-1\big\} \leq \alpha < +\infty$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.1) satisfies $\sigma_{[2,2]}(f, z_0) \leq \alpha$.*

P r o o f. Let $f(z)(\not\equiv 0)$ be an analytic solution of (1.1) in $\overline{\mathbb{C}}\backslash\{z_0\}$. By Lemma 2.2, there exists a set $E_3 \subset (0,1)$ of finite logarithmic measure such that, for all $r \notin E_3$ and $r \to 0$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j \big(1 + o(1)\big), \qquad j = 1, \ldots, k. \tag{2.2}$$

Setting

$$M_{z_0}(r) = \max_{|z_0 - z| = r} \big\{|A_j(z)|\colon j = 0, 1, \ldots, k-1\big\}. \tag{2.3}$$

Since $\max\big\{\sigma_{\log}(A_j, z_0)\colon j = 0, \ldots, k-1\big\} \leq \alpha < +\infty$, then for any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $r_0 > r > 0$, we get

$$M_{z_0}(r) \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha+\varepsilon}\right\}. \tag{2.4}$$

Now, we may rewrite (1.1) as

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \big|A_{k-1}(z)\big|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + \big|A_1(z)\big|\left|\frac{f'(z)}{f(z)}\right| + \big|A_0(z)\big|. \tag{2.5}$$

Then, by substituting (2.2) and (2.3) into (2.5), we obtain

$$\left(\frac{V_{z_0}(r)}{r}\right)^k \big|1 + o(1)\big| \leq k M_{z_0}(r) \left(\frac{V_{z_0}(r)}{r}\right)^{k-1} \big|1 + o(1)\big|. \tag{2.6}$$

From (2.4) and (2.6), it follows that

$$V_{z_0}(r) \leq kr \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha+\varepsilon}\right\}\big|1 + o(1)\big|.$$

Therefore, by Lemma 2.3, we get $\sigma_{[2,2]}(f, z_0) \leq \alpha$. □

Lemma 2.5. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\sigma_{\log}(f,z_0)=\sigma$. Then there exists a subset E_4 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0|=r\in E_4$, we have*

$$\sigma=\lim_{r\longrightarrow 0}\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}=\lim_{r\longrightarrow 0}\frac{\log T_{z_0}(r,f)}{\log\log\frac{1}{r}}$$

and for any given $\varepsilon>0$,

$$M_{z_0}(r,f)>\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\},\qquad T_{z_0}(r,f)>\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}.$$

P r o o f. By the definition of the logarithmic order in Definition 1.3, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1}<\frac{n}{n+1}r_n$ and

$$\sigma=\lim_{n\longrightarrow\infty}\frac{\log\log M_{z_0}(r_n,f)}{\log\log\frac{1}{r_n}}.$$

Then, for any given $\varepsilon>0$, there exists an $n_0\in\mathbb{N}^+$ such that for all $n\geq n_0$ and for any $r\in[\frac{n}{n+1}r_n,r_n]$, we obtain

$$\frac{\log\log M_{z_0}(r_n,f)}{\log\log\frac{1}{\frac{n}{n+1}r_n}}\leq\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}\leq\frac{\log\log M_{z_0}(\frac{n}{n+1}r_n,f)}{\log\log\frac{1}{r_n}}.$$

Since

$$\lim_{n\longrightarrow\infty}\frac{\log\log M_{z_0}(r_n,f)}{\log\log\frac{1}{\frac{n}{n+1}r_n}}=\lim_{n\longrightarrow\infty}\frac{\log\log M_{z_0}(\frac{n}{n+1}r_n,f)}{\log\log\frac{1}{r_n}}=\sigma,$$

then for any $r\in[\frac{n}{n+1}r_n,r_n]$, we get

$$\lim_{r\longrightarrow 0}\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}=\sigma. \tag{2.7}$$

Set $E_4=\bigcup\limits_{n=n_0}^{\infty}[\frac{n}{n+1}r_n,r_n]$. Then $m_l(E_4)=\sum\limits_{n=n_0}^{\infty}\int_{\frac{n}{n+1}r_n}^{r_n}\frac{dt}{t}=\sum\limits_{n=n_0}^{\infty}\log\left(1+\frac{1}{n}\right)=\infty$. From (2.7), it follows that, for any given $\varepsilon>0$,

$$M_{z_0}(r,f)>\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\}.$$

Similarly, we can also get

$$\lim_{r\longrightarrow 0}\frac{\log T_{z_0}(r,f)}{\log\log\frac{1}{r}}=\sigma$$

and for any given $\varepsilon>0$

$$T_{z_0}(r,f)>\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}.$$

□

Similarly, by using the definition of the logarithmic type in Definition 1.4, we can prove the following lemma.

Lemma 2.6. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with finite logarithmic order $1 \leq \sigma_{\log}(f, z_0) = \sigma < +\infty$ and finite logarithmic type $0 < \tau_{\log,M}(f, z_0) < +\infty$. Then there exists a subset E_5 of $(0,1)$ that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_5$, we have*

$$\tau_{\log,M}(f, z_0) = \lim_{r\longrightarrow 0} \frac{\log M_{z_0}(r, f)}{(\log \frac{1}{r})^{\sigma}}$$

and for any given $\beta < \tau_{\log,M}(f, z_0)$,

$$M_{z_0}(r, f) > \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\sigma}\right\}.$$

We use the same proof of Lemma 4 in [13], we can easily prove the following lemma for the $[p, q]$-order.

Lemma 2.7. *Let p and q be two integers with $p \geq q \geq 1$, and let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then*

$$\sigma_{[p,q]}(f', z_0) = \sigma_{[p,q]}(f, z_0).$$

Lemma 2.8 ([19]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then the following statements hold*

(i) $T_{z_0}(r, \frac{1}{f}) = T_{z_0}(r, f) + O(1)$;

(ii) $T_{z_0}(r, f') < O\big(T_{z_0}(r, f) + \log\frac{1}{r}\big), \quad r \in (0, r_0]\backslash E_6$, *where* $E_6 \subset (0, r_0]$ *with* $m_l(E_6) < \infty$.

Lemma 2.9 ([6]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let $k \in \mathbb{N}$. Then*

$$m_{z_0}\big(r, \frac{f^{(k)}}{f}\big) = O\big(T_{z_0}(r, f) + \log\frac{1}{r}\big), \quad \text{for all } r \in (0,1)\backslash E_7 \text{ with } \quad m_l(E_7) < \infty.$$

Lemma 2.10. *Let $F(z) \not\equiv 0$, $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let f be a non-constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.2) satisfying*

$$\max\left\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0) \colon (j = 0, \ldots, k-1)\right\} < \sigma_{[2,2]}(f, z_0).$$

Then $\overline{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0)$.

P r o o f. We may rewrite (1.2) as

$$\frac{1}{f(z)} = \frac{1}{F(z)}\left(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)} + A_0(z)\right). \tag{2.8}$$

By Lemma 2.8 and (2.8) we get

$$\begin{aligned} T_{z_0}(r, f) =& T_{z_0}(r, \frac{1}{f}) + O(1) \\ =& m_{z_0}(r, \frac{1}{f}) + N_{z_0}(r, \frac{1}{f}) + O(1) \\ \leq& \sum_{j=0}^{k-1} m_{z_0}(r, A_j) + \sum_{j=1}^{k} m_{z_0}\big(r, \frac{f^{(j)}}{f}\big) + m_{z_0}(r, \frac{1}{F}) + N_{z_0}(r, \frac{1}{f}) + O(1). \end{aligned} \tag{2.9}$$

From (1.2), it is easy to see that if f has a zero at z_1 of order $m\,(m > k)$, then F must have a zero at z_1 of order at least $m-k$. Hence

$$n(r, \frac{1}{f}) \le k\overline{n}(r, \frac{1}{f}) + n(r, \frac{1}{F})$$

and

$$N_{z_0}(r, \frac{1}{f}) \le k\overline{N}_{z_0}(r, \frac{1}{f}) + N_{z_0}(r, \frac{1}{F}). \tag{2.10}$$

By Lemma 2.9, there exists a set $E_7 \subset (0, r_0]$ that has a finite logarithmic measure such that for all $|z_0 - z| = r \in (0, r_0]\setminus E_7$, we obtain

$$\sum_{j=1}^{k} m_{z_0}\big(r, \frac{f^{(j)}}{f}\big) = O\big(T_{z_0}(r, f) + \log\frac{1}{r}\big) \le \frac{1}{2}T_{z_0}(r, f). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we get

$$\frac{1}{2}T_{z_0}(r, f) \le k\overline{N}_{z_0}(r, \frac{1}{f}) + T_{z_0}(r, F) + \sum_{j=0}^{k-1} T_{z_0}(r, A_j) + O(1).$$

This implies that $\sigma_{[2,2]}(f, z_0) \le \max\big\{\overline{\lambda}_{[2,2]}(f, z_0), \sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0)\colon (j = 0, \ldots, k-1)\big\}$. Since

$$\max\big\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0)\colon (j = 0, \ldots, k-1)\big\} < \sigma_{[2,2]}(f, z_0),$$

then we obtain $\sigma_{[2,2]}(f, z_0) \le \overline{\lambda}_{[2,2]}(f, z_0)$. On the other hand, by definition we have $\overline{\lambda}_{[2,2]}(f, z_0) \le$ $\le \lambda_{[2,2]}(f, z_0) \le \sigma_{[2,2]}(f, z_0)$, therefore

$$\sigma_{[2,2]}(f, z_0) = \overline{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0).$$ □

§ 3. Proof of the theorems

Proof of Theorem 1.1.

P r o o f. We assume that $f \not\equiv 0$ is an analytic solution of (1.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$. From (1.1), we have

$$|A_0(z)| \le \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.1}$$

By the hypotheses of Theorem 1.1, for real constants $0 \le \beta < \alpha$, $\mu \ge 1$ and $|z_0 - z| = r \to 0$. we have

$$|A_0(z)| \ge \exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\mu}\right\} \tag{3.2}$$

and

$$|A_j(z)| \le \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\mu}\right\}, \qquad j = 1, \ldots, k-1. \tag{3.3}$$

By Lemma 2.1, there exist a subset $E_1 \subset (0, 1)$ having finite logarithmic measure and a constant $C > 0$ that depends only on λ, such that for all $r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le C\left[\frac{1}{r}T_{z_0}(\lambda r, f)\right]^{2j}, \qquad j = 1, \ldots, k. \tag{3.4}$$

Substituting (3.2)–(3.4) into (3.1), for all $r \notin E_1$ and $r \to 0$, we obtain

$$\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\mu}\right\} \le kC\left[\frac{1}{r}T_{z_0}(\lambda r, f)\right]^{2k} \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\mu}\right\}. \tag{3.5}$$

From (3.5), it follows that

$$\exp\left\{(\alpha-\beta)\left(\log\frac{1}{r}\right)^{\mu}\right\} \le kC\left[\frac{1}{r}T_{z_0}(\lambda r, f)\right]^{2k}. \tag{3.6}$$

From (3.6), we conclude that $\sigma_{[2,2]}(f, z_0) \ge \mu - 1$. The proof is completed. □

Proof of Theorem 1.2.

P r o o f. By the hypotheses of Theorem 1.2, there exists a set $E \subset (0,1)$ of infinite logarithmic measure such that, for all $r \in E$ and $r \to 0$, (3.2) and (3.3) hold. Then, similarly as in (3.1)–(3.6) in the proof of Theorem 1.1, for all $r \in E \setminus E_1$ and $r \to 0$, we get that (3.6) holds which implies $\sigma_{[2,2]}(f, z_0) \ge \mu - 1$. □

Proof of Theorem 1.3.

P r o o f. First, by Theorem 1.2, we can obtain $\sigma_{[2,2]}(f, z_0) \ge \sigma - 1 - \varepsilon$ and since $\varepsilon > 0$ is arbitrary, we have

$$\sigma_{[2,2]}(f, z_0) \ge \sigma_{\log}(A_0, z_0) - 1 = \sigma - 1. \tag{3.7}$$

Next, we may rewrite (1.1) as

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|. \tag{3.8}$$

By the definition of $\sigma_{\log}(A_j, z_0)$ (as in Definition 1.3), for any given $\varepsilon > 0$ and $r \to 0$, we have

$$|A_j(z)| \le \exp\left\{\left(\log\frac{1}{r}\right)^{\sigma+\varepsilon}\right\}, \qquad j = 0, \dots, k-1. \tag{3.9}$$

By Lemma 2.2, there exists a set $E_3 \subset (0,1)$ of finite logarithmic measure such that, for all $r \notin E_3$ and $r \to 0$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^{j}(1 + o(1)), \qquad j = 1, \dots, k, \tag{3.10}$$

where $|f(z_r)| = M_{z_0}(r, f) = \max_{|z-z_0|=r}|f(z)|$. Substituting (3.9) and (3.10) into (3.8), we get

$$\left(\frac{V_{z_0}(r)}{r}\right)^{k}\Big|1 + o(1)\Big| \le k\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma+\varepsilon}\right\}\left(\frac{V_{z_0}(r)}{r}\right)^{k-1}\Big|1 + o(1)\Big|.$$

From this, it follows that

$$V_{z_0}(r) \le kr\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma+\varepsilon}\right\}\Big|1 + o(1)\Big|.$$

This implies that

$$\sigma_{[2,2]}(f, z_0) \le \sigma_{\log}(A_0, z_0) = \sigma. \tag{3.11}$$

From (3.7) and (3.11), we obtain

$$\sigma_{\log}(A_0, z_0) - 1 \le \sigma_{[2,2]}(f, z_0) \le \sigma_{\log}(A_0, z_0).$$ □

Proof of Theorem 1.4.

P r o o f. Set $\max\{\sigma_{\log}(A_j, z_0)\colon j \neq 0\} < \sigma_0 < \alpha < \sigma_{\log}(A_0, z_0)$. For any given $\varepsilon > 0$, there exists a $r_0 > 0$ such that for all $r_0 > r > 0$, we have

$$|A_j(z)| \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\sigma_0+\varepsilon}\right\}, \qquad j = 1, \ldots, k-1. \tag{3.12}$$

For $\sigma_0 + \varepsilon < \alpha < \sigma_{\log}(A_0, z_0)$, by Lemma 2.5, there exists a set $E_4 \subset (0, 1)$ of infinite logarithmic measure such that, for all $r \in E_4$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we have

$$|A_0(z)| > \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha}\right\}. \tag{3.13}$$

Substituting (3.4), (3.12) and (3.13) into (3.1), for all $r \in E_4 \backslash E_1$, we obtain

$$\exp\left\{\left(\log\frac{1}{r}\right)^{\alpha}\right\} \leq kC\left[\frac{1}{r}T_{z_0}(\lambda r, f)\right]^{2k} \exp\left\{\left(\log\frac{1}{r}\right)^{\sigma_0+\varepsilon}\right\}. \tag{3.14}$$

From (3.14), we get

$$\sigma_{[2,2]}(f, z_0) \geq \alpha - 1. \tag{3.15}$$

Further, by (3.15) and Lemma 2.4, we have $\alpha - 1 \leq \sigma_{[2,2]}(f, z_0) \leq \sigma_{\log}(A_0, z_0)$, which holds for each $\alpha < \sigma_{\log}(A_0, z_0)$. Thus, we obtain $\sigma_{\log}(A_0, z_0) - 1 \leq \sigma_{[2,2]}(f, z_0) \leq \sigma_{\log}(A_0, z_0)$. The proof is completed. □

Proof of Theorem 1.5.

P r o o f. Let β_0 and β be two constants such that $\max\{\tau_{\log,M}(A_j, z_0)\colon \sigma_{\log}(A_j, z_0) = \sigma, j \neq 0\} < < \beta_0 < \beta < \tau_{\log,M}(A_0, z_0)$. If $\sigma_{\log}(A_j, z_0) < \sigma_{\log}(A_0, z_0)$, then there exists $r_0 > 0$ such that for all $r_0 > r > 0$ and for any given $\varepsilon > 0$, (3.12) holds. If $\sigma_{\log}(A_j, z_0) = \sigma_{\log}(A_0, z_0)$, then, by the definition of $\tau_{\log,M}(A_j, z_0)$, for any given $\varepsilon > 0$ and for sufficiently small r, we get

$$|A_j(z)| \leq \exp\left\{\beta_0\left(\log\frac{1}{r}\right)^{\sigma}\right\}, \qquad j = 1, \ldots, k-1. \tag{3.16}$$

By Lemma 2.6, there exists a set $E_5 \subset (0, 1)$ of infinite logarithmic measure such that, for all $r \in E_5$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we obtain

$$|A_0(z)| > \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\sigma}\right\}. \tag{3.17}$$

Substituting (3.4), (3.12), (3.16) and (3.17) into (3.1), for all $r \in E_5 \backslash E_1$, we obtain

$$\exp\left\{\beta\left(\log\frac{1}{r}\right)^{\sigma}\right\} \leq kC\left[\frac{1}{r}T_{z_0}(\lambda r, f)\right]^{2k} \exp\left\{\beta_0\left(\log\frac{1}{r}\right)^{\sigma}\right\}.$$

This implies that

$$\sigma_{[2,2]}(f, z_0) \geq \sigma - 1 = \sigma_{\log}(A_0, z_0) - 1. \tag{3.18}$$

Then, by (3.18) and Lemma 2.4, we have $\sigma_{\log}(A_0, z_0) - 1 \leq \sigma_{[2,2]}(f, z_0) \leq \sigma_{\log}(A_0, z_0)$. □

Proof of Theorem 1.6.

P r o o f. Let f be an analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (1.2). Then f can be represented as

$$f(z) = C_1(z)f_1(z) + C_2(z)f_2(z) + \cdots + C_k(z)f_k(z), \tag{3.19}$$

where $f_1, f_2, \ldots, f_k$ is a solution base of (1.1) (the homogeneous equation corresponding to (1.2)) and $C_1, C_2, \ldots, C_k$ are given by the following system of equations:

$$\begin{cases} C_1'(z)f_1(z) + C_2'(z)f_2(z) + \cdots + C_k'(z)f_k(z) = 0, \\ C_1'(z)f_1'(z) + C_2'(z)f_2'(z) + \cdots + C_k'(z)f_k'(z) = 0, \\ \vdots \\ C_1'(z)f_1^{(k-1)}(z) + C_2'(z)f_2^{(k-1)}(z) + \cdots + C_k'(z)f_k^{(k-1)}(z) = F. \end{cases} \tag{3.20}$$

By (3.20), for $j = 1, \ldots, k$, we obtain

$$C_j' = F.G_j(f_1, f_2, \ldots, f_k).W(f_1, f_2, \ldots, f_k)^{-1}, \tag{3.21}$$

where

$$W(f_1, f_2, \ldots, f_k) = \begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_k(z) \\ f_1'(z) & f_2'(z) & \cdots & f_k'(z) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(k-1)}(z) & f_2^{(k-1)}(z) & \cdots & f_k^{(k-1)}(z) \end{vmatrix}$$

is the Wronksian of $f_1, f_2, \ldots, f_k$ and $G_j(f_1, f_2, \ldots, f_k)$ is differential polynomial of $f_1, f_2, \ldots, f_k$ and their derivatives with constant coefficients. From (3.21) and Lemma 2.7, for $j = 1, \ldots, k$, we get

$$\begin{aligned} \sigma_{[2,2]}(C_j, z_0) &= \sigma_{[2,2]}(C_j', z_0) \\ &\le \max\left\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(G_j(f_1, f_2, \ldots, f_k), z_0), \sigma_{[2,2]}(W(f_1, f_2, \ldots, f_k), z_0)\right\}. \end{aligned} \tag{3.22}$$

By Theorem 1.4 and the fact that $G_j(f_1, f_2, \ldots, f_k)$ and $W(f_1, f_2, \ldots, f_k)$ are both differential polynomials of $f_1, f_2, \ldots, f_k$ and their derivatives with constant coefficients, we have

$$\max\left\{\sigma_{[2,2]}(G_j(f_1, f_2, \ldots, f_k), z_0), \sigma_{[2,2]}(W(f_1, f_2, \ldots, f_k), z_0)\right\} \le \sigma_{[2,2]}(f_j, z_0) \le \sigma_{\log}(A_0, z_0). \tag{3.23}$$

By (3.19), (3.22) and (3.23), for $j = 1, \ldots, k$, we obtain

$$\begin{aligned} \sigma_{[2,2]}(f, z_0) &\le \max\left\{\sigma_{[2,2]}(f_j, z_0), \sigma_{[2,2]}(C_j, z_0)\right\} \\ &\le \max\left\{\sigma_{[2,2]}(F, z_0), \sigma_{\log}(A_0, z_0)\right\}. \end{aligned} \tag{3.24}$$

i) If $\sigma_{[2,2]}(F, z_0) \ge \sigma_{\log}(A_0, z_0)$, then from (1.2) and (3.24), we deduce that $\sigma_{[2,2]}(f, z_0) = \sigma_{[2,2]}(F, z_0)$.

ii) If $\sigma_{[2,2]}(F, z_0) < \sigma_{\log}(A_0, z_0)$, then, from (3.24), it follows that $\sigma_{[2,2]}(f, z_0) \le \sigma_{[2,2]}(A_0, z_0)$. Now, we assert that all solutions f of the equation (1.2) satisfy $\sigma_{[2,2]}(f, z_0) \ge \sigma_{\log}(A_0, z_0) - 1$ with at most one exception. In fact, if there exist two distinct analytic solutions g_1 and g_2 of (1.2) satisfying $\sigma_{[2,2]}(g_j, z_0) < \sigma_{\log}(A_0, z_0) - 1$ $(j = 1, 2)$, then $g = g_1 - g_2$ is a nonzero analytic solution of (1.1) and satisfies $\sigma_{[2,2]}(g, z_0) = \sigma_{[2,2]}(g_1 - g_2, z_0) < \sigma_{\log}(A_0, z_0) - 1$. But by Theorem 1.4, we have $\sigma_{[2,2]}(g, z_0) = \sigma_{[2,2]}(g_1 - g_2, z_0) \ge \sigma_{\log}(A_0, z_0) - 1$. This is a contradiction. Further, if f is an analytic solution of (1.2) that satisfies $\sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0)$, then

$$\max\left\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0)\colon j = 0, 1, \ldots, k-1\right\} < \sigma_{[2,2]}(f, z_0).$$

So, the assumption of Lemma 2.10 also holds and therefore $\overline{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0)$.

□

Proof of Theorem 1.7.

P r o o f. By using similar discussions as in the proof of Theorem 1.6, we obtain the assertions of Theorem 1.7. □

§ 4. Examples

Here we give some examples to illustrate the sharpness of some assertions in our theorems.

Example 4.1. For Theorem 1.4, we consider the analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$

$$f(z) = \frac{1}{(z - z_0)^{2n+1}}, \quad n \in \mathbb{N}, \tag{4.1}$$

which is a solution to the following homogeneous complex differential equation

$$f'''(z) + A_2(z)f''(z) + A_1(z)f'(z) + A_0(z)f(z) = 0, \tag{4.2}$$

where $A_0(z) = \frac{(2n+1)(2n+2)(2n+3)}{(z-z_0)^3} - \frac{(2n+1)(2n+2)(3-7i)}{(z-z_0)^2} + \frac{(2n+1)(3+7i)}{(z-z_0)}$, $A_1(z) = 3 + 7i$ and $A_2(z) = 3 - 7i$. The coefficients $A_j(z)$, $j = 0, 1, 2$, satisfy the conditions of Theorem 1.4 and

$$\max\{\sigma_{\log}(A_1, z_0), \sigma_{\log}(A_2, z_0)\} = 0 < \sigma_{\log}(A_0, z_0) = 1.$$

We see that f satisfies

$$\sigma_{\log}(A_0, z_0) - 1 = \sigma_{[2,2]}(f, z_0) = 0 \le \sigma_{\log}(A_0, z_0) = 1.$$

Example 4.2. For Theorem 1.5, we consider the analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$

$$f(z) = e^{\frac{1}{(z-z_0)^{2n+1}}}, \quad n \in \mathbb{N}.$$

Note that f is a solution to the homogeneous complex differential equation (4.2), for

$$A_0(z) = \frac{4(n+1)(2n+1)^2}{(z - z_0)^{4n+5}}, \qquad A_1(z) = -\frac{(2n+2)(2n+3)}{(z - z_0)^2}, \qquad A_2(z) = \frac{2n+1}{(z - z_0)^{2n+2}}.$$

The coefficients $A_j(z)$, $j = 0, 1, 2$, satisfy the conditions of Theorem 1.5 and

$$\max\{\sigma_{\log}(A_1, z_0), \sigma_{\log}(A_2, z_0)\} = \sigma_{\log}(A_0, z_0) = 1$$

and

$$\max\{\tau_{\log}(A_1, z_0), \tau_{\log}(A_2, z_0)\} = 2n + 2 < \tau_{\log}(A_0, z_0) = 4n + 5.$$

We remark that f satisfies

$$\sigma_{\log}(A_0, z_0) - 1 \le \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_0, z_0) = 1.$$

Example 4.3. For Theorem 1.7, the function f in (4.1) is an analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ to the following non-homogeneous linear differential equation

$$f'''(z) + A_2(z)f''(z) + A_1(z)f'(z) + A_0(z)f(z) = F(z),$$

where $A_0(z) = \frac{2n(2n+2)(2n+3)}{(z-z_0)^3}$, $A_1(z) = \frac{\sqrt{2}(2n+2)}{(z-z_0)}$, $A_2(z) = \sqrt{2}$ and $F(z) = \frac{2n(2n+2)(2n+3)}{(z-z_0)^3}$. As we see $A_j(z)$, $i = 0, 1, 2$, and $F(z)$ satisfy the conditions of Theorem 1.7 in case (ii) and

$$\max\{\sigma_{\log}(A_1, z_0), \sigma_{\log}(A_2, z_0)\} = \sigma_{\log}(A_0, z_0) = 1,$$

$$\max\{\tau_{\log}(A_1, z_0), \tau_{\log}(A_2, z_0)\} = 1 < \tau_{\log}(A_0, z_0) = 3$$

and

$$\sigma_{[2,2]}(F, z_0) = 0 < \sigma_{\log}(A_0, z_0) = 1.$$

Then f satisfies

$$\sigma_{\log}(A_0, z_0) - 1 = \sigma_{[2,2]}(f, z_0) = 0 \le \sigma_{\log}(A_0, z_0) = 1.$$

§ 5. Conclusion

In this paper, we deal with the growth properties of solutions of the following complex linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$

where $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ and $F(z)$ are analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, z_0 is an essential singular point. Since it's hard to find some general forms for the solutions of the above equations, we are interested in the study of the behavior of such solutions and specially the notion of the growth by using the concepts of logarithmic order and logarithmic type near a singular point. The strongest tool we used for establishing our results is the Nevanlinna theory which can be found in [16, 17, 22]. Under some conditions on the growth of the coefficients, we improve and extend some recent results due to Fettouch and Hamouda in [13]. Furthermore, we obtain similar results to those in [5] for the complex plane $\mathbb{C}$ case.

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А. Дахмани, Б. Белаиди

О росте решений комплексных линейных дифференциальных уравнений с аналитическими коэффициентами в $\overline{\mathbb{C}}\backslash\{z_0\}$ конечного логарифмического порядка

Ключевые слова: линейное дифференциальное уравнение, аналитическая функция, особая точка, логарифмический порядок, логарифмический тип.

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В статье изучается рост решений однородных и неоднородных комплексных линейных дифференциальных уравнений, коэффициенты которых являются аналитическими функциями в расширенной комплексной плоскости, за исключением конечной особой точки, и имеют конечный логарифмический порядок. Мы обобщаем некоторые предыдущие результаты, которые недавно получили Феттуш и Хамуда.

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