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HITTING FUNCTIONS FOR MIXED PARTITIONS

Let T_ρ be an irrational rotation on a unit circle $S^1 \simeq [0, 1)$. Consider the sequence $\{\mathcal{P}_n\}$ of increasing partitions on S^1. Define the hitting times $N_n(\mathcal{P}_n; x, y) := \inf\{j \geq 1 \mid T_\rho^j(y) \in P_n(x)\}$, where $P_n(x)$ is an element of $\mathcal{P}_n$ containing x. D. Kim and B. Seo in [9] proved that the rescaled hitting times $K_n(\mathcal{Q}_n; x, y) := \frac{\log N_n(\mathcal{Q}_n;x,y)}{n}$ a. e. (with respect to the Lebesgue measure) converge to $\log 2$, where the sequence of partitions $\{\mathcal{Q}_n\}$ is associated with chaotic map $f_2(x) := 2x \bmod 1$. The map $f_2(x)$ has positive entropy $\log 2$. A natural question is what if the sequence of partitions $\{\mathcal{P}_n\}$ is associated with a map with zero entropy. In present work we study the behavior of $K_n(\tau_n; x, y)$ with the sequence of mixed partitions $\{\tau_n\}$ such that $\mathcal{P}_n \cap [0, \frac{1}{2}]$ is associated with map f_2 and $\mathcal{D}_n \cap [\frac{1}{2}, 1]$ is associated with irrational rotation T_ρ. It is proved that $K_n(\tau_n; x, y)$ a. e. converges to a piecewise constant function with two values. Also, it is shown that there are some irrational rotations that exhibit different behavior.

Keywords: irrational rotation, hitting time, dynamical partition, limit theorem.

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Introduction

Studying the behavior of waiting and hitting times of dynamical systems is an important problem of ergodic theory. These problems are closely related to dynamical Borel–Cantelli and shrinking target problems (see, e. g., [1–8, 22]).

In present paper we study the behavior of hitting times $K_n(\mathcal{P}_n; x, y)$ depending on the type of circle partitions.

Consider a dynamical system $(M, \mathcal{F}, T, \mu)$with T-invariant probability measure μ. For a measurable subset $A \in \mathcal{F}$, $\mu(A) > 0$, we define the function $R_A\colon A \to \mathbb{N}$ as

$$R_A(x) = \min\{j \geq 1 \mid T^j x \in A\}.$$

The function R_A is called **the first return time function** to the subset of A. An important consequence of the existence of an invariant measure was obtained by A. Poincaré [23].

Theorem 1 (Poincaré's return theorem). *Let $(M, \mathcal{F}, T, \mu)$ be a dynamical system with T-invariant probability measure μ and $A \in \mathcal{F}$, $\mu(A) > 0$. Then for almost every point x of A there exists an increasing sequence $\{n_k,\ k = \overline{1, \infty}\}$, such that $T^{n_k} x \in A$.*

Poincaré's return theorem does not provide information on the times of first returns. A great number of articles have been published answering this question in various ways. Usually, the first return time of x to a partition element generated by the transformation T itself is studied.

Classical Kac's [18] lemma states that the average of the function $R_A(x)$ with respect to the invariant measure μ does not exceed unity, i. e.,

$$\int_A R_A(x) d\mu \leq 1.$$

When T is ergodic transformation, the average is equal to one.

Note that if $\{A_n\}_{n=1}^{\infty}$, $A_n \supset A_{n+1}$, $n \geq 1$, is a sequence of subsets containing x, then $R_{A_n}(x)$ is an increasing sequence. The asymptotic behavior of recurrence times was studied by many authors (see, e. g., [9, 11, 12, 14, 16–18, 20, 21]).

Let $\mathcal{P}$ be a measurable partition of M. Define a sequence of partitions $\{\mathcal{P}_n\}$ as

$$\mathcal{P}_n = \mathcal{P} \vee T^{-1}\mathcal{P} \vee T^{-2}\mathcal{P} \vee \ldots \vee T^{-n+1}\mathcal{P},$$

where $\mathcal{Q} \vee \mathcal{P} := \{Q \cap P \mid Q \in \mathcal{Q},\ P \in \mathcal{P}\}$.

Ornstein and Weiss in [13] showed that if T is ergodic transformation, then

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{n} = h(T, \mathcal{P}) \ \ \mu\text{-a. e.},$$

where $P_n(x)$ — is the element of $\mathcal{P}_n$ containing the point x. Shannon, McMillian, and Breiman in [10] proved that, if the entropy $h(T, \mathcal{P})$ is positive, then

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(\mathcal{P}_n(x))} = 1 \ \text{ a. e.}$$

Let (X, d) be a metric space, $B(x, r) = \{z \mid d(x, z) < r\}$ be a ball with radius r and center at x. Now we define the upper and lower point dimension of the measure μ to x:

$$\bar{d}_\mu(x) = \lim_{r\to 0^+} \sup \frac{\log \mu(B(x,r))}{\log r}, \quad \underline{d}_\mu(x) = \lim_{r\to 0^+} \inf \frac{\log \mu(B(x,r))}{\log r}.$$

Theorem 2 (see [9]). *Let $X \subset \mathbb{R}^m$ be a measurable subset and $T\colon X \to X$ be a measurable transformation and μ be T-invariant probability measure on X. If $\underline{d}_\mu(x) > 0$ a. e. on X then*

$$\lim_{r\to 0^+} \sup \frac{\log \mu(B(x,r))}{\log r} \leq 1 \ \textit{a. e. for measure } \mu.$$

It is well known that each irrational number $\rho \in [0, 1]$ has a unique continued fraction expansion $\rho = [k_1, k_2, \ldots, k_n, \ldots]$, where k_n are natural numbers. Denote by $\frac{p_n}{q_n} := [k_1, k_2, \ldots, k_n]$, $n \geq 1$, the convergent of ρ.

Definition 1 (see [9]). An irrational number θ, $0 < \theta < 1$, is said to be of type η if

$$\eta = \sup\{\beta \mid \lim_{n\to\infty} \inf n^\beta \|n\theta\| = 0\}.$$

For each $t \in \mathbb{R}$ we define its norm $\|\cdot\|$ by $\|t\| = \min_{n\in\mathbb{Z}} |t - n|$, i. e., the distance to the nearest integer.

Note that every irrational number is of type $\eta \geq 1$ since $q_n^{1-\varepsilon}\|q_n\theta\| < 1/q_n^{\varepsilon}$ for every $\varepsilon > 0$. Let us denote by M_η the set of all irrational numbers of type η, belonging to segment $[0, 1]$. The set of irrational numbers of type 1 has measure 1 and includes the set of irrational numbers of “bounded type” (the elements of continued fraction expansion are uniformly bounded), which is of measure 0 (see [10]). There exist numbers of type ∞, called the Liouville numbers.

We denote by μ_1 the Lebesgue measure on the circle S^1, and by μ_2 the Lebesgue measure on the torus $S^1 \times S^1$.

Kim and Seo in [9] studied the behavior of $K_n(\mathcal{Q}_n; x, y)$ w. r. t. sequence of partitions $\mathcal{Q}_n = \left\{Q_i^{(n)} := [i2^{-n},\ (i+1)2^{-n}),\ i = 0, 1, \ldots, 2^n - 1\right\}$ of the interval $[0, 1)$.

Theorem 3 (see [9]). *For an irrational rotation*

$$\liminf_{n\to\infty} K_n(\mathcal{Q}_n; x, y) = \log 2 \ \ a.\,e.$$

and

$$\limsup_{n\to\infty} K_n(\mathcal{Q}_n; x, y) = \eta \log 2 \ \ a.\,e.$$

The last theorem implies that if ρ is of type 1, then

$$\lim_{n\to\infty} K_n(\mathcal{Q}_n; x, y) = \log 2.$$

Notice that the sequence of partitions $\mathcal{Q}_n$ can be obtained by chaotic map $f_2(x) = 2x \bmod 1$. It is well known that the map f_2 has positive entropy $\log 2$.

Each irrational rotation T_ρ has zero entropy. In present paper we study the behavior of hitting times of irrational rotations when the interval $[0, 1)$ is divided into subintervals with different kind partitions. Combining the partitions $\mathcal{Q}_n^l$ and $\mathcal{D}_n^r$ of the segments $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$, respectively, we obtain a partition τ_n of the circle. Notice that the lengths of the intervals of the partition τ_n decrease exponentially to zero uniformly. Since ρ is irrational, the orbit of any point is dense on the circle. Hence, the hitting function $N_n(\tau_n; x, y)$ takes finite number of values. As n tends to infinite, the length of the intervals of the partitions τ_n decreases exponentially to zero and the maximum value of the hitting function $N_n(\tau_n; x, y)$ exponentially increases to infinity. Let us define the rescaled hitting functions $K_n(\tau_n; x, y)$:

$$K_n(\tau_n; x, y) := \frac{\log N_n(\tau_n; x, y)}{n}.$$

An important problem is to determine the asymptotic behavior of the random variable $K_n(\tau_n; x, y)$ as $n \to +\infty$.

Now we pass to formulate the main results of our work.

First we consider the irrational rotations T_ρ for “unbounded type” ρ, i. e., the sequence of elements of continued fraction expansion of ρ is unbounded.

Theorem 4. *Let T_ρ be a linear rotation of the circle and the number ρ belongs to the subset of M_1. Then, there exist subsets $A_1(\rho), A_2(\rho) \subset S^1 \times S^1$, $\mu_2(A_1) = \mu_2(A_2) = \frac{1}{2}$, such that*

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \begin{cases} \log 2, & (x, y) \in A_1, \\ \dfrac{\pi^2}{12 \log 2}, & (x, y) \in A_2. \end{cases}$$

Let ρ be a quadratic irrational number, then its continued fraction is eventually periodic:

$$\rho = [a_1, a_2, \ldots, a_m,\ k_1, k_2, \ldots, k_s,\ k_1, k_2, \ldots, k_s, \ldots].$$

Put

$$\bar{\rho} := \sqrt[s]{\rho_1 \rho_2 \ldots \rho_s},$$

where

$$\rho_1^{-1} := [k_1, k_s, k_{s-1}, \ldots, k_1, \ldots],$$
$$\rho_2^{-1} := [k_2, k_1, k_s, k_{s-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_1, \ldots],$$
$$\ldots,$$
$$\rho_s^{-1} := [k_s, k_{s-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_1, \ldots].$$

Now we formulate the limit theorem concerning the case of irrational rotations of quadratic type.

Theorem 5. *Let T_ρ be a linear rotation of the circle. Suppose ρ is a quadratic irrational number and its continued fraction expansion has the form*

$$\rho = [a_1, a_2, \dots, a_m,\ k_1, k_2, \dots, k_s,\ k_1, k_2, \dots, k_s, \dots], \quad s \geq 1.$$

Then, there are subsets $A_1(\rho), A_2(\rho) \subset S^1 \times S^1$, $\mu_2(A_1) = \mu_2(A_2) = \frac{1}{2}$, such that

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \begin{cases} \log 2, & (x,y) \in A_1, \\ \log \bar{\rho}, & (x,y) \in A_2. \end{cases}$$

Remark 1. The results of Theorems 4 and 5 can be extended to mixed partition on $[0, b]$ and $[b, 1]$, respectively, i. e., in statements of Theorems 4 and 5 the point $\frac{1}{2}$ should be replaced by b.

§ 1. Dynamical partitions and first return functions

We consider the linear irrational rotation $T_\rho x = (x + \rho) \bmod 1$ on the circle $S^1 = \mathbb{R}^1/\mathbb{Z}^1 \simeq \simeq [0, 1)$. Define the positive orbit of the point as $\mathcal{O}^+(x_0) = \{x_n \mid x_n = T_\rho^n x_0,\ n \geq 0\}$. Consider the first return times q_n of the irrational number ρ. We put $x_0 = 0$. Using the points of the orbit $\mathcal{O}^+(x_0)$, we can define a sequence of dynamical partitions $\{\mathcal{D}_n(x_0),\ n \geq 1\}$ (see [15]). For every $n \geq 1$ the piece of the orbit $\mathcal{O}_n(x_0) = \{x_i = T_\rho^i x_0,\ 0 \leq i < q_n + q_{n-1}\}$ will divide the circle into $q_{n-1} + q_n$ closed intervals. The obtained partition of a circle we denote by $\mathcal{D}_n(x_0)$.

Let us describe the partitions $\mathcal{D}_n(x_0)$, $n \geq 1$. For even n the point $x_{q_n} = T_\rho^{q_n} x_0$ lies to the right of x_0, and for odd n it lies to the left of x_0. Let us denote by $\Delta_0^{(n)}$ the interval connecting the points x_0 and x_{q_n}. Put $\Delta_i^{(n)} = T_\rho^i \Delta_0^{(n)}$, $i \geq 0$. The intervals $\Delta_s^{(n)}$, $s \geq 0$, are called segments of the nth rank. It is well known (see [23]) that the partition $\mathcal{D}_n(x_0)$ consists of the intervals of two ranks:

$$\mathcal{D}_n(x_0) = \left\{\Delta_0^{(n-1)}, \Delta_1^{(n-1)}, \dots, \Delta_{q_n-1}^{(n-1)}\right\} \bigcup \left\{\Delta_0^{(n)}, \Delta_1^{(n)}, \dots, \Delta_{q_{n-1}-1}^{(n)}\right\}.$$

The partition $\mathcal{D}_n$ is called the n**th dynamical partition of circle.** Passing from partition $\mathcal{D}_n(x_0)$ to $\mathcal{D}_{n+1}(x_0)$, all intervals $\Delta_i^{(n)}$, $0 \leq i < q_{n-1}$, of nth rank are preserved, but each interval $\Delta_i^{(n-1)}$, $0 \leq i < q_n$, of $(n-1)$th rank is divided into $(k_{n+1} + 1)$ intervals:

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)} \cup \bigcup_{j=0}^{k_{n+1}-1} \Delta_{i+jq_{n-1}}^{(n)}. \tag{1}$$

In fact, we get a decreasing sequence of dynamical partitions of the circle

$$\mathcal{D}_1(x_0) \preceq \mathcal{D}_2(x_0) \preceq \dots \mathcal{D}_n(x_0) \preceq \mathcal{D}_{n+1}(x_0) \preceq \dots.$$

Set $\Delta_n := |\Delta_i^{(n)}(x_0)|$, $i \geq 0$, where $|\cdot|$ denotes the length of interval. The structure of dynamical partition $\mathcal{D}_n(x_0)$ implies $q_{n+1}\Delta_n + q_n\Delta_{n+1} = 1$. Moreover, the relation (1) implies that the sequence of lengths of intervals $\{\Delta_n,\ n \geq 1\}$ satisfies the following difference equation:

$$\Delta_{n-1} = k_{n+1}\Delta_n + \Delta_{n+1}, \quad n \geq 1. \tag{2}$$

Now we give the well known properties of Δ_n (see, e. g., [10]):

$$\Delta_n := |q_n\rho - p_n|, \quad \frac{1}{2q_{n+1}} < \Delta_n < \frac{1}{q_{n+1}}, \quad n \geq 1. \tag{3}$$

Now, using orbits of two different points x_0 and z_0, we define another decreasing sequence $\{\mathcal{D}_n(x_0; z_0),\ n \geq 1\}$ of dynamic partitions of the circle. Let $0 < z_0 < \rho$. There exists $n_0 := n(z_0)$ such that

$$\Delta_{n_0+1} < |[0, z_0]| = z_0 \leq \Delta_{n_0}.$$

Using the equation (2), we get that there is a unique number $k_0 = k_0(z_0)$, $0 \leq k_0 \leq k_{n+1} - 1$, such that

$$\Delta_{n+2} + k_0\Delta_{n+1} < z_0 \leq \Delta_{n+2} + (k_0 + 1)\Delta_{n+1}.$$

For an even n_0, using the structure of dynamical partitions $\mathcal{D}_{n_0}(x_0)$ and the arrangement of points $z_i = T_\rho^i z_0$, $i \leq 0$, on the circle, it can easily be shown that $0 = x_0 < z_{-q_{n_0}-k_0 q_{n_0+1}} < x_{-q_{n_0+1}} < z_0$. Let us introduce the following notations:

$$L_0^{(n_0)} := [x_0; z_{-q_{n_0}-k_0 q_{n_0+1}}), \quad R_0^{(n_0)} := [x_{-q_{n_0+1}}; z_0), \quad M_0^{(n_0)} := [z_{-q_n - k^0 q_{n_0+1}}; x_{-q_{n_0+1}}).$$

If n_0 is odd then

$$R_0^{(n_0)} := [x_0; z_{-q_{n_0+1}}), \quad L_0^{(n_0)} := [x_{-q_{n_0}-k_0 q_{n+1}}; z_0), \quad M_0^{(n_0)} := [z_{-q_{n_0+1}}; x_{-q_{n_0}-k_0 q_{n_0+1}}).$$

By construction, the semiintervals $L_0^{(n_0)}$, $M_0^{(n_0)}$ and $R_0^{(n_0)}$ are pairwise disjoint and

$$[x_0; z_0) = L_0^{(n_0)} \cup M_0^{(n_0)} \cup R_0^{(n_0)}.$$

Lemma 1. *For every $n_0 \geq 1$ the system of intervals*

$$\mathcal{D}_{n_0}(x_0; z_0) = \left\{T_\rho^i L_0^{(n_0)},\ 0 \leq i < q_{n_0} + k_0 q_{n_0+1},\ T_\rho^j R_0^{(n_0)}\right\} \bigcup$$
$$\bigcup \left\{0 \leq j < q_{n_0+1},\ T_\rho^r M_0^{(n_0)},\ 0 \leq r < q_{n_0} + (k_0+1) q_{n_0+1}\right\}$$

forms a partition of the circle.

P r o o f. Using the structure of dynamic partitions we obtain that if n_0 is even and $0 \leq j < q_{n_0+1}$, then

$$\Delta_j^{(n_0)}(x_0) = \bigcup_{s=1}^{k_0-1} \left(L_{j+q_{n_0}+s q_{n_0+1}}^{(n_0)} \cup M_{j+q_{n_0}+s q_{n_0+1}}^{(n_0)}\right) \cup \left(L_j^{(n_0)} \cup M_j^{(n_0)} \cup R_j^{(n_0)} \cup M_{j+q_{n_0}+k_0 q_{n_0+1}}^{(n_0)}\right),$$
$$\Delta_i^{(n_0+1)}(x_0) = L_{i+q_{n_0+1}}^{(n_0)} \cup M_{i+q_{n_0+1}}^{(n_0)}, \quad 0 \leq i < q_{n_0}.$$

If n_0 is odd and $0 \leq j < q_{n_0+1}$, then

$$\Delta_j^{(n_0)}(z_0) = \bigcup_{s=1}^{k_0-1} \left(L_{j+q_{n_0}+s q_{n_0+1}}^{(n_0)} \cup M_{j+q_{n_0}+s q_{n_0+1}}^{(n_0)}\right) \cup \left(L_j^{(n_0)} \cup M_j^{(n_0)} \cup R_j^{(n_0)} \cup M_{j+q_{n_0}+k_0 q_{n_0+1}}^{(n_0)}\right),$$
$$\Delta_i^{(n_0+1)}(z_0) = L_{i+q_{n_0+1}}^{(n_0)} \cup M_{i+q_{n_0+1}}^{(n_0)}, \quad 0 \leq i < q_{n_0}.$$

Obviously, the system of intervals

$$\left\{T_\rho^i \Delta_0^{(n_0)},\ 0 \leq i < q_{n_0+1},\ T_\rho^j \Delta_0^{(n_0+1)},\ 0 \leq j < q_{n_0}\right\}$$

will be a partition of the circle. This means that the system of segments $\mathcal{D}_{n_0}(x_0; z_0)$ forms a partition of the circle. Lemma 1 is proved. $\square$

Let $A \subset S^1$ be a measurable subset with $\mu(A) > 0$. Define the **first return time** $R_A \colon A \to \mathbb{N}$ as

$$R_A(x) = \inf\{j \geq 1 \mid T_\rho^j x \in A\} \text{ for all } x \in A.$$

Slater in [19] proved that in the case when A is an interval, $R_A(x)$ takes only two or three values. The short proof of this fact was found by D. Kim and B. Seo in [9].

Theorem 6 (see [9]). *Let T_ρ be a linear irrational rotation and $b \in (0, \rho)$ be a fixed number. Let $n_b = n(b) \geq 0$ be an integer such that $\Delta_{n_b+1} < b \leq \Delta_{n_b}$ and $K_b := K(b)$ be an integer defined as $K_b = \max\{k \geq 0 \mid k\Delta_{n_b+1} + b < \Delta_{n_b}\}$. If $n_b \geq 0$ is odd, then*

$$R_{[0,b)}(x) = \begin{cases} q_{n_b+1}, & x \in [0, b - \Delta_{n_b+1}), \\ q_{n_b} + (K_b + 2)q_{n_b+1}, & x \in [b - \Delta_{n_b+1}, \Delta_{n_b} - (K_b + 1)\Delta_{n_b+1}), \\ q_{n_b} + (K_b + 1)q_{n_b+1}, & x \in [\Delta_{n_b} - (K_b + 1)\Delta_{n_b+1}, b). \end{cases}$$

If $n_b \geq 0$ is even, then

$$R_{[0,b)}(x) = \begin{cases} q_{n_b} + (K_b + 1)q_{n_b+1}, & x \in [0, b + (K_b + 1)\Delta_{n_b+1} - \Delta_{n_b}), \\ q_{n_b} + (K_b + 2)q_{n_b+1}, & x \in [b + (K_b + 1)\Delta_{n_b+1} - \Delta_{n_b}, \Delta_{n_b+1}), \\ q_{n_b+1}, & x \in [\Delta_{n_b+1}, b). \end{cases}$$

Remark 2. Note that the value $R_{[0,b)}(x)$ at the middle interval is the sum of two other values.

Remark 3. $0 \leq K_b < k_{n+1}$ for all $n_b \geq 0$.

§ 2. Proof of Theorems 4 and 5

It is well known that continued fractions are closely related to the Gauss mapping and have many remarkable ergodic properties. We first present three necessary lemmas from the ergodic theory of continued fractions and then prove Theorems 4 and 5 (see [23]).

Lemma 2 (see [23]). *There is a subset M of irrational numbers $M \subset [0, 1)$ such that $\mu_1(M) = 1$, and for every $\rho \in M$:*

$$\lim_{n\to\infty} \frac{\log q_n(\rho)}{n} = \frac{\pi^2}{12 \log 2}.$$

Now we will prove several lemmas and then move to prove Theorem 4.

Lemma 3. *Let $\rho = [a_1, a_2, \ldots, a_m, \ k_1, k_2, \ldots, k_s, \ k_1, k_2, \ldots, k_s, \ldots]$ be a quadratic irrational number. For every $j = \overline{1, s}$ there is the following limit:*

$$\lim_{n\to\infty} \log \frac{q_{ns+j}}{q_{ns+j-1}} = \log \rho_j.$$

P r o o f. Using the relation $q_{n+1} = k_{n+1}q_n + q_{n-1}$, we obtain:

$$\frac{q_{ns+j}}{q_{ns+j-1}} = \frac{k_j q_{ns+j-1} + q_{ns+j-2}}{q_{ns+j-1}} = k_j + \cfrac{1}{\cfrac{q_{ns+j-1}}{q_{ns+j-2}}} =$$

$$= k_j + \cfrac{1}{k_{j-1} + \cfrac{q_{ns+j-2}}{q_{ns+j-3}}} = \ldots = k_j + \cfrac{1}{k_{j-1} + \cfrac{1}{\ldots a_2 + \cfrac{1}{a_1}}} =$$

$$= [k_j, k_{j-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_j, k_{j-1}, \ldots, k_1, \ldots, k_j, k_{j-1}, \ldots, k_1, a_m, a_{m-1}, \ldots, a_1].$$

This implies the assertion of Lemma 3. $\square$

Let us formulate the next lemma.

Lemma 4. *Let* $\rho = [a_1, a_2, \ldots, a_m,\ k_1, k_2, \ldots, k_s,\ k_1, k_2, \ldots, k_s, \ldots]$ *be a quadratic irrational number. Then for every* $1 \le i \le s$ *there is a limit*

$$\lim_{n\to\infty} \frac{\log q_{ns+i}}{ns+i} = \log \bar{\rho}, \tag{4}$$

and its value does not depend on j.

P r o o f. Using the Stolz–Cesaro theorem and Lemma 3, we obtain:

$$\begin{aligned}\lim_{n\to\infty} \frac{\log q_{ns+i}}{ns+i} &= \lim_{n\to\infty} \frac{\log q_{ns+i} - \log q_{(n-1)s+i}}{s} \\ &= \lim_{n\to\infty} \frac{1}{s}\Big(\log \frac{q_{ns+i}}{q_{ns+i-1}} + \log \frac{q_{ns+i-1}}{q_{ns+i-2}} + \ldots + \log \frac{q_{(n-1)s+i+1}}{q_{(n-1)s+i}}\Big) \\ &= \frac{\log \rho_1 + \log \rho_2 + \ldots + \log \rho_s}{s} \\ &= \log \bar{\rho}.\end{aligned}$$

Lemma 4 is proved. □

One has the following lemma.

Lemma 5. *For almost all irrational numbers* $\rho \in [0, 1)$ *and almost all points (w. r. t. Lebesgue measure)* $(x, y) \in [0, \frac{1}{2}) \times [0, 1)$ *there is a limit*

$$\limsup_{n\to\infty} K_n(\tau_n; x, y) = \log 2.$$

P r o o f. It is well known that a linear irrational rotation of the circle T_ρ is ergodic. Let $x \in [0, \frac{1}{2})$. Using the assertion of Theorem 6, we obtain

$$N_n(\tau_n; x, y) = R_{Q_s^{(n)}}(y) \le q_{j(n)} + (k+1)q_{j(n)+1}, \quad 1 \le k \le k_{j(n)+1},$$

$$K_n(\tau_n; x, y) = \frac{1}{n} \log N_n(\tau_n; x, y) \le \frac{\log(q_{j(n)} + (k+1)q_{j(n)+1})}{n} \le \frac{\log 2 + \log q_{j(n)+2}}{n},$$

where $j(n)$ can be determined uniquely from relation $\Delta_{j(n)+1} \le 2^{-n} < \Delta_{j(n)}$.

Assume that the irrational number ρ is of type 1. For sufficiently small $\varepsilon > 0$ the following estimate holds

$$q_{j(n)+2}^{1+\varepsilon} \Delta_0^{(j(n)+2)} > C_\varepsilon,$$

where C_ε is a positive constant.

Using the inequality (3), we get

$$q_{j(n)+2} < \frac{1}{\Delta_{j(n)+1}} \quad \text{and} \quad q_{j(n)+1}^{1+\varepsilon} > \frac{C_\varepsilon}{\Delta_{j(n)+1}}.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty}\frac{\log(2q_{j(n)+2})}{n}\le&\lim_{n\to\infty}\frac{\log 2-\log\Delta_{j(n)+1}}{n}\\ \le&\lim_{n\to\infty}\frac{\log 2-\log C_\varepsilon+(1+\varepsilon)\log q_{j(n)+1}}{n}\\ \le&\lim_{n\to\infty}\frac{\log 2-\log C_\varepsilon-(1+\varepsilon)\log\Delta_{(j(n))}}{n}\\ \le&\lim_{n\to\infty}\frac{\log 2-\log C_\varepsilon-(1+\varepsilon)\log 2^{-n}}{n}\\ =&(1+\varepsilon)\log 2.\end{aligned}$$ □

Next we prove the following useful lemma.

Lemma 6. *For Lebesgue almost all $y\in[0,1)$ and an arbitrary point $x\in[0,\frac{1}{2})$, the following limit holds:*

$$\lim_{n\to\infty}\inf K_n(\tau_n;x,y)=\log 2.$$

P r o o f. Consider the sequence of sets

$$G_n^\varepsilon=\left\{(x,y)\,\middle|\,x\in\left[0,\frac{1}{2}\right),\ K_n(\tau_n;x,y)<\log 2-\varepsilon\right\}.$$

Now we estimate $\mu_2(G_n^\varepsilon)$. It is clear that

$$\begin{aligned}\mu_2(G_n^\varepsilon)=&\mu_2\left(\left\{(x,y)\,\middle|\,x\in\left[0,\frac{1}{2},\right),\ R_{Q_n(x)}(y)<\exp\{n(\log 2-\varepsilon)\}\right\}\right)\\ =&\sum_{Q_n(x)\in\mathcal{Q}_n}\sum_{r=1}^{\left[e^{n(\log 2-\varepsilon)}\right]}|Q_n(x)|\mu_1\left(\{y\,|\,R_{Q_n(x)}(y)=r\}\right)\\ \le&\sum_{Q_n(x)\in\mathcal{Q}_n}\sum_{r=1}^{\left[e^{n(\log 2-\varepsilon)}\right]}|Q_n(x)|\cdot|Q_n(x)|\\ =&\sum_{Q_n(x)\in\mathcal{Q}_n}e^{n(\log 2-\varepsilon)}|Q_n(x)|^2=\frac{2^{-n}}{2}e^{n(\log 2-\varepsilon)}=\frac{e^{-n\varepsilon}}{2}.\end{aligned}$$

Using the last equality, we obtain

$$\mu_2(G_n^\varepsilon)<\frac{e^{-n\varepsilon}}{2}.$$

Since the series $\sum\mu_2(G_n^\epsilon)$ converges, using the Borel–Cantelli lemma, we get that

$$\mu_2(\limsup G_n^\epsilon)=0.$$

The last relation together with the definition of $G_n^\varepsilon(x)$ implies that there exists an increasing sequence $\{n_k\}$ such that $\lim\limits_{k\to\infty}K_{n_k}(\tau_{n_k};x,y)=\log 2.$ □

Using Lemmas 5 and 6, we get that, for almost all irrational numbers ρ,

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \log 2 \quad \mu_2\text{-a. e. } (x, y) \in \left[0, \frac{1}{2}\right) \times S^1.$$

We denote by A_1 the subset of $\left[0, \frac{1}{2}\right) \times S^1$ where the last limit exists.

Next we prove the following useful lemma.

Lemma 7. *For almost all irrational numbers $\rho \in [0, 1)$ and almost all points $x \in \left(\frac{1}{2}, 1\right)$:*

$$\limsup_{n\to\infty} K_n(\tau_n; x, y) = \frac{\pi^2}{12 \log 2}.$$

P r o o f. Let $x \in \Delta_k^{(n)}$, $0 \le k < q_{n+1}$. We have $N_n(\tau_n; x, y) = R_{\Delta_k^{(n)}}(y)$. For $x \in \Delta_k^{(n)}$, using Theorem 6, we obtain

$$N_n(\tau_n; x, y) \le q_n + q_{n+1},$$

and

$$K_n(\tau_n; x, y) = \frac{1}{n} \log N_n(\tau_n; x, y) \le \frac{\log(q_n + q_{n+1})}{n} \le \frac{\log(2q_{n+1})}{n}.$$

Using the assertion of Lemma 2, we get that, for almost all irrational numbers $\rho \in (0, 1)$,

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2},$$

respectively, for almost all irrational numbers

$$\limsup_{n\to\infty} K_n(\tau_n; x, y) = \lim_{n\to\infty} \frac{\log 2q_{n+1}}{n} = \frac{\pi^2}{12 \log 2}.$$

In the case $x \in \Delta_j^{(n+1)}$, $0 \le j < q_n$, the last relation can be proved similarly. □

Lemma 8. *For almost all $y \in [0, 1)$ and an arbitrary point $x \in \left(\frac{1}{2}, 1\right)$, the following limit holds:*

$$\liminf_{n\to\infty} K_n(\tau_n; x, y) = \frac{\pi^2}{12 \log 2}.$$

P r o o f. Let $x \in \left(\frac{1}{2}, 1\right)$. Put

$$C_n^{\varepsilon}(x) := \left\{ (x, y) \,\middle|\, x \in \left(\frac{1}{2}, 1\right), K_n(\tau_n; x, y) < \frac{\pi^2}{12 \log 2} - \varepsilon \right\}$$
$$= \left\{ (x, y) \,\middle|\, R_{\Delta^{(n)}(x)}(y) < e^{-n\varepsilon + \frac{n\pi^2}{12 \log 2}} \right\},$$

where $\Delta^{(n)}(x)$ is an element of the partition $\mathcal{D}_n^r$ containing x. Now let us estimate the probabilities of the events C_n^ε:

$$\begin{aligned}\mu_2(C_n^\varepsilon) &= \sum_{l=1}^{\left[e^{-n\varepsilon+\frac{n\pi^2}{12\log 2}}\right]} \mu_2(\{(x,y) \mid R_{\Delta^{(n)}(x)}(y) = l\}) \\ &\le \sum_{\Delta^{(n)}(x)\in\mathcal{D}_n^r} e^{-n\varepsilon+\frac{n\pi^2}{12\log 2}} |\Delta^{(n)}(x)| \\ &< \frac{\max\{|\Delta^{(n)}(x)|\}}{2} e^{-n\varepsilon+\frac{n\pi^2}{12\log 2}} \\ &< \frac{e^{-n\varepsilon+\frac{n\pi^2}{12\log 2}}}{q_n}.\end{aligned}$$

By Lemma 2, for $\forall\varepsilon > 0\ \exists N_0 = N_0(\varepsilon)$ such that, for all $\forall n > N_0$,

$$\frac{n\pi^2}{12\log 2} - \frac{\varepsilon}{2} < \frac{\log q_n}{n} < \frac{n\pi^2}{12\log 2} + \frac{\varepsilon}{2}.$$

Using the last inequalities, it can be easily shown that

$$q_n > \exp\left\{n\left(\frac{n\pi^2}{12\log 2} - \frac{\varepsilon}{2}\right)\right\},$$

and

$$\mu_2(C_n^\varepsilon) < e^{-\frac{n\varepsilon}{2}}.$$

Obviously, the series $\sum \mu_2(C_n^\varepsilon)$ converges. Using the Borel–Cantelli lemma, we get that

$$\mu_2(\limsup C_n^\varepsilon) = 0,$$

which implies that there is a sequence $\{n_k\}$ such that

$$\lim_{n_k\to\infty} K_{n_k}(\tau_{n_k}; x, y) = \frac{\pi^2}{12\log 2}. \qquad \square$$

Let $x \in \left(\frac{1}{2}, 1\right)$. Using Lemmas 7 and 8, we get that, for almost all irrational numbers ρ and almost all $(x,y) \in \left(\frac{1}{2}, 1\right) \times [0,1)$,

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \frac{\pi^2}{12\log 2}.$$

We denote by A_2 the subset of $\left(\frac{1}{2}, 1\right) \times S^1$ where the last limit exists. It is easy to check that $A_2 \in S^1 \times S^1$ and $\mu_2(A_2) = \dfrac{1}{2}$.

Using Lemmas 5–8, we get that, for Lebesgue almost all irrational numbers $0 < \rho < 1$,

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \begin{cases} \log 2, & (x,y) \in A_1, \\ \dfrac{\pi^2}{12\log 2}, & (x,y) \in A_2. \end{cases}$$

Theorem 4 is proved. □

Now let us prove Theorem 5.

Let $\rho = [a_1, a_2, \ldots, a_m,\ k_1, k_2, \ldots, k_s,\ k_1, k_2, \ldots, k_s, \ldots]$ be a quadratic irrational number. It is clear that the type of a quadratic irrational number is 1. Using Lemmas 5 and 6, we get:

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \log 2 \text{ a. e. } \left[0, \frac{1}{2}\right) \times S^1.$$

We need the following Lemma.

Lemma 9. *Let T_ρ be a linear rotation of the circle and ρ be a quadratic irrational number. Then*

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \log \bar{\rho} \ \text{ a. e. } (x, y) \in \left(\frac{1}{2}, 1\right) \times S^1.$$

P r o o f. Suppose $x \in \left(\frac{1}{2}, 1\right)$. Then, there are n and l such that $x \in \Delta_l^{(n)}$. In this case, we have

$$N_n(\tau_n; x, y) = R_{\Delta_l^{(n)}}(y) \le q_n + q_{n+1} < 2q_{n+1}.$$

The relation (4) implies that

$$\lim_{n\to\infty} K_n(\tau_n; x, y) \le \lim_{n\to\infty} \frac{\log 2q_{n+1}}{n} = \lim_{n\to\infty} \frac{\log 2 + \log q_{n+1}}{n} = \log \bar{\rho}.$$

Put

$$H_n^\varepsilon = \left\{ (x, y) \,\middle|\, x \in \left(\frac{1}{2}, 1\right),\ K_n(\tau_n; x, y) < \log \bar{\rho} \right\}.$$

Now we estimate $\mu_2(H_n^\varepsilon)$. It is clear that

$$\begin{aligned}
\mu_2(H_n^\varepsilon) =& \mu_2\left(\left\{ (x, y) \,\middle|\, x \in \left(\frac{1}{2}, 1\right),\ R_{\Delta_n(x)}(y) < \exp\left\{n(\log \bar{\rho} - \varepsilon)\right\} \right\}\right) \\
=& \sum_{\Delta_n(x) \in \mathcal{D}_n^r} \sum_{r=1}^{\left[e^{n(\log \bar{\rho} - \varepsilon)}\right]} |\Delta_n(x)| \mu_1\left(\left\{ y \,\middle|\, R_{\Delta_n(x)}(y) = r \right\}\right) \\
\le& \sum_{\Delta_n(x) \in \mathcal{D}_n^r} \sum_{r=1}^{\left[e^{n(\log \bar{\rho} - \varepsilon)}\right]} |\Delta_n(x)| \cdot |\Delta_n(x)| \\
=& \sum_{\Delta_n(x) \in \mathcal{D}_n^r} e^{n(\log \bar{\rho} - \varepsilon)} |\Delta_n(x)|^2 \\
\le& \frac{1}{2} \max |\Delta_n(x)| e^{n(\log \bar{\rho} - \varepsilon)} \\
=& \frac{e^{n(\log \bar{\rho} - \varepsilon)}}{2} |\Delta_0^{(n)}| \\
<& q_{n+1}^{-1} e^{n(\log \bar{\rho} - \varepsilon)}.
\end{aligned}$$

By (4), for $\forall \varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that, for all $n > N_0$,

$$e^{n\left(\log \bar{\rho} - \frac{\varepsilon}{2}\right)} < q_{n+1} < e^{n\left(\log \bar{\rho} + \frac{\varepsilon}{2}\right)}.$$

Using the last equality, we obtain:

$$\mu_2(H_n^{\varepsilon}) < e^{\frac{-n\varepsilon}{2}}.$$

Since the series $\sum \mu_2(H_n^{\epsilon}))$ converges, using the Borel–Cantelli lemma, we get that

$$\mu_2(\limsup G_n^{\epsilon}) = 0.$$

Hence, it follows that

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \log \bar{\rho} \ \text{ a. e. } (x, y) \in \left(\frac{1}{2}, 1\right) \times S^1. \qquad \square$$

Using the statements of Lemmas 7–9, we obtain:

$$\lim_{n\to\infty} K_n(\tau_n; x, y) = \begin{cases} \log 2, & (x, y) \in A_1, \\ \log \bar{\rho}, & (x, y) \in A_2. \end{cases}$$

Theorem 5 is completely proved. $\square$

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Функции попадания для смешанных разбиений окружности

Ключевые слова: иррациональное вращение, время попадания, динамическое разбиение, предельная теорема.

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Пусть T_ρ — иррациональный поворот на единичной окружности $S^1 \simeq [0,1)$. Рассмотрим последовательность $\{\mathcal{P}_n\}$ возрастающих разбиений на S^1. Определим время попадания $N_n(\mathcal{P}_n; x, y) := \inf\{j \geq 1 \mid T_\rho^j(y) \in P_n(x)\}$, где $P_n(x)$ — элемент разбиения $\mathcal{P}_n$, содержащий точку x. Д. Ким и Б. Сео [9] доказали, что время попадания $K_n(\mathcal{Q}_n; x, y) := \frac{\log N_n(\mathcal{Q}_n; x, y)}{n}$ почти всюду (по мере Лебега) сходится к $\log 2$, где последовательность разбиений $\{\mathcal{Q}_n\}$ порождена хаотическим отображением $f_2(x) := 2x \bmod 1$. Хорошо известно, что отображение f_2 имеет положительную энтропию $\log 2$. Возникает естественный вопрос о том, что если последовательность разбиений $\{\mathcal{P}_n\}$ порождена отображением с нулевой энтропией. В настоящей работе мы изучаем поведение $K_n(\tau_n; x, y)$ с последовательностью смешанных разбиений τ_n таких, что $\mathcal{Q}_n \cap [0, \frac{1}{2}]$ порождена отображением f_2, а $\mathcal{D}_n \cap [\frac{1}{2}, 1]$ порождена иррациональным поворотом T_ρ. Доказано, что $K_n(\tau_n; x, y)$ почти всюду (по мере Лебега) сходится к кусочно-постоянной функции с двумя значениями. Также показано, что существуют некоторые иррациональные повороты, демонстрирующие различное поведение.

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