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NON-LINEAR WAVE PROPAGATION IN A WEAKLY COMPRESSIBLE KELVIN–VOIGT LIQUID CONTAINING BUBBLY CLUSTERS

The effect of bubble–bubble interaction on wave propagation in homogeneous weakly compressible viscoelastic bubbly flow is investigated using the reductive perturbation method. The bubble dynamics equation is derived using the kinetic energy conservation approach. The bubble dynamics and mixture equations are coupled with the equation of state for gas to investigate the shock wave propagation phenomenon in the mixture. A two-dimensional Korteweg–de Vries–Burger (KdVB) equation in terms of a pressure profile is derived. It is found that the bubble–bubble interaction has no effect when using the parameters under our consideration.

Keywords: shock wave, Kelvin–Voigt liquid, bubbly liquid, KdVB equation.

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Introduction

The mixture of liquid with gas bubbles has been observed in many areas of science and engineering [1, 2]. A homogeneous bubbly liquid flow model in which mixture equations are coupled with the bubble dynamics equation and that of state for gas has been considered in [3–5].

A modified Rayleigh–Plesset equation used to describe the bubble dynamics in a compressible viscoelastic liquid has been studied in [6, 7]. Other characteristics of bubble dynamics in cavitating flow and their effects such as relative motion between bubbles and liquid, bubble–bubble interaction, thermal and mass diffusion are important in order to understand fully the behavior of bubbles in cavitation flows [4, 8]. These studies are focused on the single bubble dynamics behavior. Analysis of a bubbly compressible viscoelastic liquid accounting for bubble–bubble interaction is scarce [9–12].

The effect of variation of a bubble radius on the shock wave propagation in bubbly liquid flow was investigated by Ando [13, 14]. To analyse the effect of randomly distributed bubbles on shock wave propagation, non-uniform bubble distribution in a homogeneous bubbly liquid was investigated in [15, 16]. Furthermore, the effect of a spatially non-uniform bubble distribution on the propagation of shock wave with high void fraction was considered in [1].

Using the homogeneity assumption, Kubota et al. [17] incorporated the velocity potential of other bubbles within a cluster to model the bubble–bubble interaction, which is a simplification of works by Chahine [18, 19]. Delale et al. [20, 21] carried out direct numerical simulations to shock propagation in bubbly viscous liquids with bubble–bubble interaction, but their approach is computationally expensive and handles small number of bubbles. A comprehensive reviews on various modeling strategies adopted to capture the response of bubble clusters to pressure fluctuation was carried out in [22, 23]. In the above-mentioned studies, bubbles are assumed to be compressible with clusters in only viscous liquid, no account is given for the corresponding behavior in a viscoelatic liquid, whereas in reality most bubbly mixtures are viscoelastic [1, 10, 11, 22–25]. No analysis on shock wave propagation in a weakly compressible viscoelastic liquid containing bubbly clusters is conducted.

The objective of the present work is to investigate the shock wave propagation in a bubbly weakly compressible viscoelastic liquid, including the bubble–bubble interaction effect in continuum homogeneous flows through the derivation of a non-linear evolution equation using the perturbation method. This method has been well adopted and used to analyse many multi-phase, bubbly liquid flow problems [26,27].

As an extension to the earlier studies by Delale et al. [20,21], we study shock wave propagation in weakly compressible viscoelastic liquid flow with bubble–bubble interaction effects taken into account. For the analysis of bubbly shock flows, we consider the solution of the unsteady, homogeneous bubbly mixture model [28–30]. In this study therefore, we exclude thermal damping and relative motion between bubbles and liquid. A modified version of the Rayleigh–Plesset equation in compressible viscoelastic liquid is proposed by incorporating the effect of bubble–bubble interactions. The bubble dynamics equation will be derived using conservation of kinetic energy.

§ 1. Formulation of the problem

We consider a homogeneous bubbly weakly compressible viscoelastic liquid mixture with averaged density, velocity and pressure. The number of gas bubbles in the unit mass of mixture is constant N. The pressure in gas bubble is uniform. There are no formation and destruction of bubbles. The bubbles are assumed to be spherical and develop only radial motion. There are no mass and heat transfer between the bubbles and viscoelastic liquid. The viscoelastic influence of the liquid is considered only at the interphase boundary between the liquid and gas bubbles. In these assumptions, the system of equations for the description of waves propagation in weakly compressible viscoelastic liquid with gas bubbles will be derived.

1.1 Equations of the mixture

The effective equations for mixture flow using the above assumptions are governed by the continuity and momentum equations, which are written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \qquad \rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p, \tag{1.1}$$

where $\mathbf{V}(x,y,t) = (v^{(1)}(x,y,t), v^{(2)}(x,y,t))$ is the averaged velocity of the liquid, ∇ is the "del" operator vector in x and y coordinates and t is time. ρ is the averaged density of the bubbly-liquid mixture, p is the averaged pressure of the mixture, the extra stress effect of the liquid is negligible and therefore it is omitted. Equation (1.1) is derived in [31,32].

The mixture density is a combination of liquid and gas densities expressed as

$$\rho = \rho_l (1-\alpha) + \alpha \rho_g, \quad \alpha = V\rho, \quad V = \frac{4}{3}\pi N R^3, \tag{1.2}$$

where $R = R(x,y,t)$ is the bubble radius, ρ_l, ρ_g are densities of the liquid and gas bubbles respectively. α is the gas volume fraction in the unit mass of the mixture. Equation (1.2) is expressed by the following relations [32]:

$$\rho = \frac{\rho_l}{(1 - X + V\rho_l)}, \quad X = V\rho_g. \tag{1.3}$$

We suppose that

$$R(x,y,t) = R_0 + \phi(x,y,t), \quad |\phi(x,y,t)| < R_0, \tag{1.4}$$

where $\phi(x,y,t)$ is the perturbation of bubble radius from the equilibrium. The series expansion of (1.3) to order ϕ^2 is

$$\rho = \rho_0 - \gamma_1\phi + \gamma_2\phi^2 - \cdots,$$
$$\rho_0 = \frac{\rho_l}{(1 - X + V_0\rho_l)}, \quad \gamma_1 = \frac{3V_0\rho_l^2}{R_0(1 - X + V_0\rho_l)^2},$$
$$\gamma_2 = \frac{6V_0\rho_l^2\,(X - 1 + 2V_0\rho_l)}{R_0^2(1 - X + V_0\rho_l)^3}, \quad V_0 = \frac{4}{3}\pi N R_0^3. \tag{1.5}$$

1.2 Equation of motion for bubbles in compressible Kelvin–Voigt fluid

In this research, spherical bubble dynamics equation in a compressible viscoelastic liquid is derived using energy conservation approach [33,34]. It is shown that the compressibility corrections appear only in the radial equation.

Let the velocity vector in the surrounding bubble $\mathbf{V} = v_r e_r$ be defined by the wave equation [35]

$$\Delta\varphi - \frac{1}{c_l^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{1.6}$$

where φ is the velocity potential, $v = \nabla\varphi$. c_l is the speed of sound in the liquid, approximated by its value in the unperturbed liquid ($c_l \gg 1$). The potential describing the radial motion of the bubble and liquid satisfying (1.6) is of the form

$$\varphi = \frac{1}{r}A\left(t - \frac{r}{c_l}\right), \tag{1.7}$$

where r is distance from the bubble centre. Taking the series expansion of (1.7), we have

$$\varphi(R,t) = \frac{A(t)}{R} - \frac{1}{c_l}\frac{\partial A(t)}{\partial t} + O\left(c_l^{-2}\right), \quad r = R. \tag{1.8}$$

With the assumption of no mass transfer across the liquid–bubble boundary, the wall velocity at the liquid–bubble boundary is simply the time rate of change of the radius, therefore at the bubble boundary, A is obtained using the kinematic boundary condition

$$\mathbf{V}\cdot\mathbf{n} = \frac{\partial\varphi}{\partial r} = R', \quad r = R(t). \tag{1.9}$$

This is due to the fact that in the near field, the mixture behaves in an incompressible fashion. The derivative of (1.7) with respect to r, is

$$\frac{\partial\varphi}{\partial r} = -\frac{1}{c_l r}A'\left(t - \frac{r}{c_l}\right) - \frac{1}{r^2}A\left(t - \frac{r}{c_l}\right), \tag{1.10}$$

where the Taylor series expansion of (1.10) is

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=R(t)} = -\frac{A(t)}{R^2} + O\left(c_l^{-2}\right). \tag{1.11}$$

Using (1.11), equation (1.9) becomes

$$A = -R^2\,R' + O\left(c_l^{-2}\right), \tag{1.12}$$

or

$$v_r = \frac{R^2 R'}{r^2}. \tag{1.13}$$

Let the kinetic energy conservation of the liquid given in [33, 34] be

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}} \rho \mathbf{V}\cdot\mathbf{V}\, d\mathcal{V} = \int_{\mathcal{V}} p\nabla\cdot\mathbf{V}\, d\mathcal{V} - \int_{\mathcal{S}_\mathcal{B}+\mathcal{S}_\infty} p\mathbf{V}\cdot\mathbf{n}\, d\mathcal{S}, \tag{1.14}$$

where $\mathbf{n}$ is the outward unit normal, and $\mathcal{V}$ is the volume comprised between the bubble surface $\mathcal{S}_\mathcal{B}$ and a concentric spherical surface $\mathcal{S}_\infty$. The kinetic energy conservation approach is on the basis that the work done on a liquid by all forces acting on the liquid is equal to the change of the kinetic energy of the liquid. The conservation of momentum for a spherically symmetric radial flow yields [36]

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\frac{\partial \mathbf{V}}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{2}{r}(\tau_{rr} - \tau_{\theta\theta}), \tag{1.15}$$

where τ_{rr} and $\tau_{\theta\theta}$ are the normal stress components in the radial and polar directions respectively. The boundary and initial conditions are

$$p = p_\infty, \quad r = \infty, \quad R = R_0, \quad R' = 0 \quad \text{at} \quad t = 0,$$

where p_∞ is the pressure far from the bubble, R_0 and R' are the equilibrium and velocity of the bubble radius. The dot indicates time derivative.

To derive the bubble dynamics in compressible viscoelastic fluid, we shall account for the compressibility of the surrounding medium to first-order acoustic approximation. With the definition of the velocity potential, the momentum equation (1.15) may be integrated once with respect to the radial coordinate, from $r \to \infty$. Assuming that the velocity potential vanishes as $r \to \infty$, we have

$$-\rho\left(\frac{\partial \varphi}{\partial t} + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) = p - p_\infty + \int_r^\infty \left(\frac{\partial \tau_{rr}}{\partial r} + \frac{2}{r}(\tau_{rr} - \tau_{\theta\theta})\right) dr. \tag{1.16}$$

From the fact that $\tau_{rr} = -2\tau_{\theta\theta}$ [37], equation (1.16) becomes

$$p = p_\infty - \rho\left(\frac{\partial \varphi}{\partial t} + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) + \tau_{rr}(R) - \int_R^\infty \frac{3\tau_{rr}}{r}\, dr. \tag{1.17}$$

Using (1.17) in the first term on the right-hand side of (1.14), we have

$$\begin{aligned}\int_{\mathcal{V}} p\nabla\cdot\mathbf{V}\, d\mathcal{V} = &-\int_{\mathcal{V}} \rho\left(\frac{\partial \varphi}{\partial t} + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right)\nabla\cdot\mathbf{V}\, d\mathcal{V} \\ &+ \int_{\mathcal{V}}\left(p_\infty + \tau_{rr}(R) - \int_R^\infty \frac{3\tau_{rr}}{r}\, dr\right)\nabla\cdot\mathbf{V}\, d\mathcal{V}.\end{aligned} \tag{1.18}$$

Using (1.6), equation (1.18) becomes

$$\begin{aligned}\int_{\mathcal{V}} p\nabla\cdot\mathbf{V}\, d\mathcal{V} = &-\frac{\rho}{c_l^2}\int_{\mathcal{V}} \frac{\partial \varphi}{\partial t}\frac{\partial^2 \varphi}{\partial t^2}\, d\mathcal{V} - \frac{\rho}{2c_l^2}\int_{\mathcal{V}} \mathbf{V}\cdot\mathbf{V}\frac{\partial^2 \varphi}{\partial t^2}\, d\mathcal{V} \\ &+ \int_{\mathcal{V}}\left(p_\infty + \tau_{rr}(R) - \int_R^\infty \frac{3\tau_{rr}}{r}\, dr\right)\nabla\cdot\mathbf{V}\, d\mathcal{V}.\end{aligned} \tag{1.19}$$

Applying divergence theorem, (1.19) becomes

$$\int_{\mathcal{V}} p\nabla\cdot\mathbf{V}\,d\mathcal{V} = -\frac{\rho}{c_l^2}\int_{\mathcal{V}}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t^2}\,d\mathcal{V} - \frac{\rho}{2c_l^2}\int_{\mathcal{V}}\mathbf{V}\cdot\mathbf{V}\frac{\partial^2\varphi}{\partial t^2}\,d\mathcal{V} + \int_{\mathcal{S}_{\mathcal{B}}+\mathcal{S}_\infty}\left(p_\infty + \tau_{rr}(R) - \int_R^\infty \frac{3\tau_{rr}}{r}\,dr\right)\mathbf{V}\cdot\mathbf{n}\,d\mathcal{S}. \tag{1.20}$$

Simplifying the first term on the right-hand side of (1.20), we have

$$\begin{aligned}\frac{1}{c_l^2}\int_{\mathcal{V}}\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial t^2}\,d\mathcal{V} &= \frac{1}{2c_l^2}\int_{\mathcal{V}}\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t}\right)^2 d\mathcal{V}\\ &= \frac{2\pi}{c_l^2}\int_R^\infty\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}\right)^2 dr\\ &= -\frac{2\pi}{c_l}\int_R^\infty\frac{\partial}{\partial r}\left(\frac{\partial A}{\partial t}\right)^2 dr.\end{aligned} \tag{1.21}$$

Equation (1.21) was obtained by using $A(t - r/c_l)$ and $\partial A/\partial t = -c_l\,\partial A/\partial r$, and that $d\mathcal{V} = 4\pi r^2\,dr$. Using (1.11) and (1.21), equation (1.20) becomes

$$\int_{\mathcal{V}} p\nabla\cdot\mathbf{V}\,d\mathcal{V} = \int_{\mathcal{S}_{\mathcal{B}}+\mathcal{S}_\infty}\left(p_\infty + \tau_{rr}(R) - \int_R^\infty\frac{3\tau_{rr}}{r}\,dr\right)\mathbf{V}\cdot\mathbf{n}\,d\mathcal{S} - \frac{2\pi\rho}{c_l}\left(\frac{\partial A}{\partial t}\right)^2_{r=R(t)} + O\left(c_l^{-2}\right). \tag{1.22}$$

Let the right hand side of (1.14) be

$$\mathcal{A} = \int_{\mathcal{V}} p\nabla\cdot\mathbf{V}\,d\mathcal{V} - \int_{\mathcal{S}_{\mathcal{B}}+\mathcal{S}_\infty} p\mathbf{V}\cdot\mathbf{n}\,d\mathcal{S}. \tag{1.23}$$

Substituting (1.22) into (1.23) we have

$$\begin{aligned}\mathcal{A} &= \int_{\mathcal{S}_{\mathcal{B}}}\left(p_\infty - p_B + \tau_{rr}(R) - \int_R^\infty\frac{3\tau_{rr}}{r}\,dr\right)\mathbf{V}\cdot\mathbf{n}\,d\mathcal{S} - \frac{2\pi\rho}{c_l}\left(\frac{\partial A}{\partial t}\right)^2_{r=R(t)} + O\left(c_l^{-2}\right)\\ &= 4\pi R^2R'\left(p_B - p_\infty - \tau_{rr}(R) + \int_R^\infty\frac{3\tau_{rr}}{r}\,dr\right) - \frac{2\pi\rho}{c_l}R^2(2R'^2 + RR'')^2.\end{aligned} \tag{1.24}$$

Calculating the left-hand side of (1.14),

$$\begin{aligned}\mathcal{B} = \frac{1}{2}\int_{\mathcal{V}}\rho\mathbf{V}\cdot\mathbf{V}\,d\mathcal{V} &= \frac{1}{2}\rho\int_{\mathcal{V}}\nabla\varphi\cdot\nabla\varphi\,d\mathcal{V}\\ &= \frac{1}{2}\rho\int_{\mathcal{V}}\left(\nabla\cdot(\varphi\nabla\varphi) - \varphi\nabla^2\varphi\right)d\mathcal{V}\\ &= -2\pi\rho R^2\varphi(R,t)R' - \frac{\rho}{2c_l^2}\int_{\mathcal{V}}\varphi\frac{\partial^2\varphi}{\partial t^2}\,d\mathcal{V}.\end{aligned} \tag{1.25}$$

Using (1.7),

$$\begin{aligned}\frac{1}{c_l^2}\int_{\mathcal{V}}\varphi\frac{\partial^2\varphi}{\partial t^2}\,d\mathcal{V} &= \frac{4\pi}{c_l^2}\int_R^\infty A\frac{\partial^2 A}{\partial t^2}\,dr = -\frac{4\pi}{c_l}\int_R^\infty A\frac{\partial^2 A}{\partial r\partial t}\,dr\\ &= -\frac{4\pi}{c_l}\int_R^\infty\left[\frac{\partial}{\partial r}\left(A\frac{\partial A}{\partial t}\right) - \frac{\partial A}{\partial r}\frac{\partial A}{\partial t}\right]dr\\ &= \frac{4\pi}{c_l}\left(A\frac{\partial A}{\partial t}\right)_{r=R(t)} - \frac{4\pi}{c_l^2}\int_R^\infty\left(\frac{\partial A}{\partial t}\right)^2 dr.\end{aligned} \tag{1.26}$$

The Taylor series expansion of the non-integral term on the right-hand side of (1.26) is

$$\frac{1}{c_l}\left(A\frac{\partial A}{\partial t}\right)_{r=R(t)} = \frac{1}{c_l}A\frac{\partial A}{\partial t} + \left(c_l^{-2}\right). \tag{1.27}$$

Letting $s = t - R/c_l$, and assuming that the motion starts at $t = 0$ so that $A(s)$ vanishes for $s < 0$, the integral term on the right-hand side of (1.26) is

$$\frac{1}{c_l^2}\int_R^\infty \left(\frac{\partial A}{\partial t}\right)^2 dr = \frac{1}{c_l}\int_0^{t-R/c_l}\left(\frac{\partial A}{\partial s}\right)^2 ds = \frac{1}{c_l}\int_0^t \left(\frac{\partial A}{\partial s}\right)^2 ds + O\left(c_l^{-2}\right). \tag{1.28}$$

Using (1.27) and (1.28) into (1.26), we have

$$\frac{1}{c_l^2}\int_{\mathcal{V}} \varphi\frac{\partial^2\varphi}{\partial t^2}\, d\mathcal{V} = \frac{4\pi}{c_l}A\frac{\partial A}{\partial t} - \frac{4\pi}{c_l}\int_0^t \left(\frac{\partial A}{\partial s}\right)^2 ds + O\left(c_l^{-2}\right). \tag{1.29}$$

Substituting (1.8) and (1.29) into (1.25), we have

$$\mathcal{B} = -2\pi\rho R^2\left(\frac{A(t)}{R} - \frac{1}{c_l}\frac{\partial A(t)}{\partial t}\right)R' - \left(\frac{2\pi\rho}{c_l}A\frac{\partial A}{\partial t} - \frac{2\pi\rho}{c_l}\int_0^t \left(\frac{\partial A}{\partial s}\right)^2 ds + O\left(c_l^{-2}\right)\right),$$

and using (1.12),

$$\mathcal{B} = 2\pi\rho\left[-RR'f(t) + \frac{1}{c_l}\frac{\partial A(t)}{\partial t}(R^2R' - A) + \frac{1}{c_l}\int_0^t \left(\frac{\partial A}{\partial s}\right)^2 ds + O\left(c_l^{-2}\right)\right]. \tag{1.30}$$

By substituting (1.12) into (1.30), we have

$$\mathcal{B} = 2\pi\rho\left[R^3R'^2 - \frac{2}{c_l}R^3R'(2R'^2 + RR'') + \frac{1}{c_l}\int_0^t \left(\frac{\partial A}{\partial s}\right)^2 ds + O\left(c_l^{-2}\right)\right]. \tag{1.31}$$

The derivative of (1.31) with respect to time is

$$\frac{d\mathcal{B}}{dt} = \frac{1}{2}\frac{d}{dt}\int_{\mathcal{V}} \rho\mathbf{V}\cdot\mathbf{V}\, d\mathcal{V} = 4\pi\rho\mathcal{C}, \tag{1.32}$$

where

$$\begin{aligned}\mathcal{C} =& \frac{3}{2}R^2R'^3 + R^3R'R'' - \frac{(2R'^3 + 6RR'R'' + R^2R''')R^2R'}{c_l}\\ &- \frac{2(2RR'^2 + R^2R'')RR'^2}{c_l} - \frac{(2RR'^2 + R^2R'')^2R''}{c_l} - \frac{(2RR'^2 + R^2R'')^2}{2c_l}.\end{aligned}$$

Equating (1.24) and (1.32), we have

$$\left(\frac{3}{2}R'^2 + RR''\right) - \frac{1}{c_l}(2R'^3 + 6RR'R'' + R^2R''') = \frac{1}{\rho_l}\left[p_B - p_\infty - \tau_{rr}(R) + \int_R^\infty \frac{3\tau_{rr}}{r}\, dr\right], \tag{1.33}$$

wherein the near field $\rho = \rho_l$ is constant. Equation (1.33) is the single bubble dynamics equation in compressible viscoelastic liquid.

We shall adopt the model for bubble clusters used in [17, 38] to account for bubble–bubble interactions in a bubbly viscoelastic liquid. The interactions are considered within spherical clusters, each of radius R_0. The total pressure on bubble i due to radiation from the neighbouring bubbles within the cluster at the origin 0 is

$$p_i = \frac{\rho_l}{4\pi}\sum_i \frac{1}{r_i}\mathbf{V}\left(t - \frac{r_i}{c_l}\right), \tag{1.34}$$

where r_i is the distance between bubble i and bubble at the centre O [39]. Taking the series expansion of (1.34), we have

$$p_i = \frac{\rho_l}{4\pi}\left(\frac{V''(t)}{r_i} - \frac{V'''(t)}{c_l}\right) + O\big(c_l^{-2}\big).$$

After differentiating $V(t) = \frac{4}{3}\pi R(t)^3$ once and recognizing that $R_i = R$ (by the homogeneity of the mixture),

$$p_i = \rho_l\frac{d}{dt}\left(\sum_i \frac{1}{r_i}R'R^2\right) - \frac{\rho_l}{c_l}\frac{d^2}{dt^2}(R'R^2), \qquad R < r_i,$$

assuming that all bubbles within the spherical cluster have the same radius. The Rayleigh–Plesset equation (1.33) is now modified to

$$\begin{aligned}&\frac{d}{dt}\left(\sum_i \frac{1}{r_i}R'R^2\right) - \frac{l}{c_l}\frac{d^2}{dt^2}(R'R^2) + \left(\frac{3}{2}R'^2 + RR''\right) - \frac{1}{c_l}(2R'^3 + 6RR'R'' + R^2R''')\\ &= \frac{1}{\rho_l}\left[p_B - p_\infty - \tau_{rr}(R) + \int_R^\infty \frac{3\tau_{rr}}{r}\,dr\right].\end{aligned} \tag{1.35}$$

Let $\frac{4}{3}Nk^3$ be the number of bubbles within the cluster of radius k, where N is constant. The summation on the left-hand side in (1.35) becomes [17]

$$\frac{d}{dt}\left(\sum_i \frac{1}{r_i}R'R^2\right) = 2\pi k^2N[R^2R'' + 2RR'^2]. \tag{1.36}$$

Substituting (1.36) into (1.35), we have

$$\begin{aligned}&[2\pi k^2NR + 1]RR'' + \left[4\pi k^2NR + \frac{3}{2}\right]R'^2 - \frac{2}{c_l}(2R'^3 + 6RR'R'' + R^2R''')\\ &= \frac{1}{\rho_l}\left[p_B - p_\infty - \tau_{rr}(R) + \int_R^\infty \frac{3\tau_{rr}}{r}\,dr\right].\end{aligned} \tag{1.37}$$

This is the modified bubble dynamics equation in a compressible viscoelastic liquid with bubble–bubble interaction. In the case $c_l \to \infty$, equation (1.37) becomes the bubble dynamics equation in an incompressible viscoelastic liquid [41] with bubble–bubble interaction. The relation between the pressure inside the bubble and the pressure in the liquid involves both the surface tension and viscoelastic stresses, given as [36]

$$p_B = p_g - \frac{2\sigma}{R} + \tau_{rr}(R), \qquad r = R, \tag{1.38}$$

where σ is the surface tension between the gas bubble and viscoelastic liquid. Using (1.38) in (1.37), we have

$$\begin{aligned}&[2\pi k^2NR + 1]RR'' + \left[4\pi k^2NR + \frac{3}{2}\right]R'^2 - \frac{2}{c_l}(2R'^3 + 6RR'R'' + R^2R''')\\ &= \frac{1}{\rho_l}\left[p_g - \frac{2\sigma}{R} - p_\infty + \int_R^\infty \frac{3\tau_{rr}}{r}\,dr\right].\end{aligned} \tag{1.39}$$

For the measure of the cluster radius k, we use the concept adopted in [38] that is $k = \Lambda R_0 = 300R_0$. To describe the thermal effects that occur during the evolution of the bubble, a relationship between the gas pressure p_g and bubble radius R is considered using the polytropic equation for gas

$$p_g = p_{g0}\left(\frac{R_0}{R}\right)^{3n}, \qquad p_{g0} = p_0 + \frac{2\sigma}{R_0},$$

where n denotes the polytropic index; $n = 1$ implies that bubble behaves isothermally, whereas $1 < n \leq 1.4$ represents the isentropic condition of the gas [4], p_{g0} is the pressure of the gas in an equilibrium state. Kelvin–Voigt fluid is the viscoelastic liquid that would be considered in our analysis, the stress tensor $(\boldsymbol{\tau})$ is given as [40, 42, 43]

$$\tau_{rr} = 2(G\gamma_{rr} + \mu\dot{\gamma}_{rr}), \qquad \dot{\gamma} = \mathbf{D} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & 0 & 0 \\ 0 & \dfrac{u_r}{r} & 0 \\ 0 & 0 & \dfrac{u_r}{r} \end{pmatrix}, \tag{1.40}$$

where $\dot{\gamma}_{rr}$ is the rate of strain tensor, $\mathbf{D}$ is the symmetric velocity gradient tensor. By symmetry, the velocity of the bubble has only radial component $v = v(r,t)$ and the stress tensor is diagonal; $\boldsymbol{\tau} = \operatorname{diag}(\tau_{rr}, \tau_{\theta\theta}, \tau_{\phi\phi})$, μ is the dynamic viscosity of the liquid. From (1.40), using (1.13), we have

$$\dot{\gamma}_{rr} = \frac{\partial u_r}{\partial r} = -\frac{2R^2R'}{r^3}, \qquad \gamma_{rr} = \int_{R_0}^{R} \dot{\gamma}_{rr}\,dt = -\frac{2(R^3 - R_0^3)}{3r^3}. \tag{1.41}$$

Using (1.41) in (1.40), we have

$$\tau_{rr} = -\frac{4G(R^3 - R_0^3)}{3r^3} - \frac{4\mu R^2R'}{r^3}. \tag{1.42}$$

Using (1.42), the integral term in (1.39) becomes

$$\int_R^\infty \frac{3\tau_{rr}}{r}\,dr = -\frac{4}{3}\frac{G(R^3 - R_0^3)}{R^3} - \frac{4\mu R'}{R},$$

and subsequently (1.39) becomes

$$\begin{aligned}&[2\pi k^2NR + 1]RR'' + \left[4\pi k^2NR + \frac{3}{2}\right]R'^2 - \frac{2}{c_l}(2R'^3 + 6RR'R'' + R^2R''') \\ &= \frac{1}{\rho_l}\left[\left(p_0 + \frac{2\sigma}{R_0}\right)\left(\frac{R_0}{R}\right)^{3n} - \frac{2\sigma}{R} - p - \frac{4\mu R'}{R} - \frac{4G}{3}\left(1 - \frac{R_0^3}{R^3}\right)\right].\end{aligned} \tag{1.43}$$

If R in the modified Rayleigh–Plesset equation (1.43) is to be considered as a function of position and time; $R = R(x,y,t)$ to take the effect of the liquid flow coordinates, therefore it gives

$$\begin{aligned}&[2\pi k^2NR + 1]RR_{tt} + \left[4\pi k^2NR + \frac{3}{2}\right]R_t^2 - \frac{2}{c_l}(2R_t^3 + 6RR_tR_{tt} + R^2R_{ttt}) \\ &= \frac{1}{\rho_l}\left[\left(p_0 + \frac{2\sigma}{R_0}\right)\left(\frac{R_0}{R}\right)^{3n} - \frac{2\sigma}{R} - p - \frac{4\mu R_t}{R} - \frac{4G}{3}\left(1 - \frac{R_0^3}{R^3}\right)\right].\end{aligned} \tag{1.44}$$

Substituting (1.4) and (1.5) into (1.1) and (1.44) to order ϕ^2, we obtain

$$-\gamma_1 \frac{\partial \phi}{\partial t} + 2\gamma_2 \phi \frac{\partial \phi}{\partial t} + \rho_0 \nabla \cdot \mathbf{V} - \gamma_1 \phi \nabla \cdot \mathbf{V} - \gamma_1 \mathbf{V} \cdot \nabla \phi = 0, \tag{1.45}$$

$$\rho_0 \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) - \gamma_1 \phi \frac{\partial \mathbf{V}}{\partial t} + \nabla p = 0, \tag{1.46}$$

$$\begin{aligned} p = &- a_1 \phi - a_2 \phi_t + a_{10} \phi \phi_t - a_3 \phi_{tt} + a_4 \phi_{ttt} \\ &- a_5 \phi_t^2 - a_6 \phi \phi_{tt} + a_7 \phi_t \phi_{tt} + a_8 \phi \phi_{ttt} + a_9 \phi^2, \end{aligned} \tag{1.47}$$

where

$$a_1 = \left[\frac{3n}{R_0} \left(p_0 + \frac{2\sigma}{R_0} \right) - \frac{2\sigma}{R_0^2} + \frac{4G}{R_0} \right], \quad a_2 = \frac{4\mu}{R_0}, \quad a_{10} = \frac{4\mu}{R_0^2},$$

$$a_3 = [2N\pi R_0^2 k^2 \rho_l + \rho_l R_0], \quad a_4 = \frac{2\rho_l R_0^2}{c_l},$$

$$a_5 = [4N\pi R_0 k^2 \rho_l + \frac{3}{2}\rho_l], \quad a_6 = [\rho_l + 4\rho_l \pi N k^2 R_0], \quad a_7 = \frac{4\rho_l R_0}{c_l},$$

$$a_8 = \frac{12\rho_l R_0}{c_l}, \quad a_9 = \left[\frac{3n(3n+1)}{2R_0^2} \left(p_0 + \frac{2\sigma}{R_0} \right) - \frac{2\sigma}{R_0^3} + \frac{8G}{R_0^2} \right].$$

Equations (1.45), (1.46) and (1.47) shall be considered for our study of wave dispersion and dissipation in the bubbly Kelvin–Voigt liquid flow.

§ 2. Linear Wave

Considering linear cases of (1.45)–(1.47), we have the following system of equations

$$\gamma_1 \frac{\partial \varphi}{\partial t} - \rho_0 \nabla \cdot \mathbf{V} = 0, \tag{2.1a}$$

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} + \nabla p = 0, \tag{2.1b}$$

$$p = -a_1 \phi - a_2 \phi_t - a_3 \phi_{tt} + a_4 \phi_{ttt}. \tag{2.1c}$$

From (2.1), we have a two-dimensional wave equation in $\phi(x, y, t)$, written in the form

$$\phi_{tt} - c_0^2 \Delta \phi = 0, \tag{2.2}$$

where $c_0^2 = a_1/\gamma_1$ is the speed of wave in the mixture. The wave equation (2.2) is linear and non-dispersive, which has a well known solution [44]

$$\phi = F(\mathbf{k} \cdot \mathbf{x} - wt).$$

For free oscillation of the bubble, the natural frequency of (2.1c) is

$$w_0 = \sqrt{\frac{a_1 a_3}{a_3^2 + a_2 a_4}}. \tag{2.3}$$

Ignoring the effect of the damping term and bubble–bubble interaction, (2.3) becomes

$$w_* = \sqrt{\frac{1}{\rho_l R_0} \left[\frac{3n}{R_0} \left(p_0 + \frac{2\sigma}{R_0} \right) - \frac{2\sigma}{R_0^2} + \frac{4G}{R_0} \right]}. \tag{2.4}$$

Equation (2.4) without the modulus of elasticity is called the Minnert natural frequency [45].

We introduce the following dimensionless variables

$$t=\frac{l}{c_0}t^*,\quad x=lx^*,\quad y=ly^*,\quad \rho=\rho_0\rho^*,\quad \varphi=R_0\varphi^*,\quad \mathbf{V}=c_0\mathbf{V}^*,\quad p=p_0p^*, \tag{2.5}$$

where the quantities l/c_0 and l are the characteristic time and length scales of the problem. Substituting (2.5) into equations (1.45)–(1.47), we have the following non-dimensional equations (dropping the asterisk)

$$\frac{\partial\phi}{\partial t}+\phi\nabla\cdot\mathbf{V}+\mathbf{V}\cdot\nabla\phi-\frac{2\gamma_2}{\gamma_1}R_0\phi\frac{\partial\phi}{\partial t}-\frac{\rho_0}{\gamma_1R_0}\nabla\cdot\mathbf{V}=0, \tag{2.6a}$$

$$\frac{\rho_0}{\gamma_1R_0}(\mathbf{V}_t+\mathbf{V}\cdot\nabla\mathbf{V})-\phi\mathbf{V}_t+\frac{p_0}{\gamma_1c_0^2R_0}\nabla p=0, \tag{2.6b}$$

$$\begin{aligned}p=&-a_{11}\phi-a_{12}\phi_t+a_{12}\phi\phi_t-a_{13}\phi_{tt}+a_{14}\phi_{ttt}\\&-a_{15}\phi_t^2-a_{16}\phi\phi_{tt}+a_{17}\phi_t\phi_{tt}+a_{18}\phi\phi_{ttt}+a_{19}\phi^2,\end{aligned} \tag{2.6c}$$

where the coefficients a_i $(i=11,\dots,19)$ are obtained as

$$\begin{aligned}
a_{11}&=\frac{R_0}{p_0}\left[\frac{3n}{R_0}\left(p_0+\frac{2\sigma}{R_0}\right)-\frac{2\sigma}{R_0^2}+\frac{4G}{R_0}\right],\quad a_{12}=\frac{4\mu w_0}{p_0},\\
a_{13}&=\frac{R_0w_0^2}{p_0}[2N\pi R_0^2k^2\rho_l+\rho_lR_0],\quad a_{14}=\frac{2\rho_l}{c_l}\frac{w_0^3R_0^3}{p_0},\\
a_{15}&=\frac{R_0^2w_0^2}{p_0}[4N\pi R_0k^2\rho_l+\frac{3}{2}\rho_l],\quad a_{16}=\frac{R_0^2w_0^2}{p_0}[\rho_l+4\rho_l\pi Nk^2R_0],\\
a_{17}&=\frac{4\rho_lw_0^3R_0^3}{c_lp_0},\quad a_{18}=\frac{12\rho_lw_0^3R_0^3}{c_lp_0},\\
a_{19}&=\left[\frac{3n(3n+1)}{2R_0^2}\left(p_0+\frac{2\sigma}{R_0}\right)-\frac{2\sigma}{R_0^3}+\frac{8G}{R_0^2}\right]\frac{R_0^2}{p_0}.
\end{aligned} \tag{2.7}$$

The linearised cases of (2.6) are simplified to

$$\frac{\partial^2\phi}{\partial t^2}-\Delta\phi-\frac{a_{12}}{a_{11}}\Delta\phi_t-\frac{a_{13}}{a_{11}}\Delta\phi_{tt}+\frac{a_{14}}{a_{11}}\Delta\phi_{ttt}=0. \tag{2.8}$$

Let the harmonic solution be of the form

$$\phi(\mathbf{x},t)=e^{i(\mathbf{k}\cdot\mathbf{x}-wt)},\quad \mathbf{k}=(k_x,k_y),\quad \mathbf{x}=(x,y), \tag{2.9}$$

where $\mathbf{k}$ is the wave number of x and y directions, while $\mathbf{x}$ is the coordinate vector of the dispersive wave; w is the angular frequency of the wave. Substituting (2.9) into (2.8), neglecting wave number k_y, we have

$$i\frac{a_{14}}{a_{11}}w^3k_x^2+i\frac{a_{12}}{a_{11}}wk_x^2+\frac{a_{13}}{a_{11}}w^2k_x^2-k_x^2+w^2=0,$$

and

$$w^2=\frac{k_x^2(-ia_{14}w^3-ia_{12}w+a_{11})}{k_x^2a_{13}+a_{11}}, \tag{2.10}$$

which is similar to the dispersion relation obtained in [46] if compressibility and viscoelasticity of the liquid are neglected. We observe from (2.10) that the linear case indicates the existence of two travelling waves in opposite directions whose phase speed is

$$c_p = \frac{w}{k_x} \approx 1 - \frac{a_{13}}{2a_{11}} k_x^2. \tag{2.11}$$

It is noted from the coefficients in equation (2.7), that the dispersive wave is influenced by the equilibrium pressure of the liquid, the initial bubbles radius, density, elasticity of the viscoelastic liquid and surface tension between the liquid and the bubble. Similarly, the damping of the wave is influenced by the equilibrium pressure of the liquid, the initial bubbles radius and density, viscosity and the compressibility of the liquid. In component form, equations (2.6a) and (2.6b) can be written as

$$\frac{\partial \phi}{\partial t} + \phi\big(v_x^{(1)} + v_y^{(2)}\big) + v^{(1)}\phi_x + v^{(2)}\phi_y - \frac{2\gamma_2 R_0}{\gamma_1}\phi\frac{\partial \phi}{\partial t} - \frac{\rho_0}{\gamma_1 R_0}\big(v_x^{(1)} + v_y^{(2)}\big) = 0, \tag{2.12}$$

$$\frac{\rho_0}{\gamma_1 R_0}\big(v_t^{(1)} + v^{(1)}v_x^{(1)} + v^{(2)}v_y^{(1)}\big) - \phi v_t^{(1)} + \frac{p_0}{\gamma_1 c_0^2 R_0} p_x = 0, \tag{2.13}$$

$$\frac{\rho_0}{\gamma_1 R_0}\big(v_t^{(2)} + v^{(1)}v_x^{(2)} + v^{(2)}v_y^{(2)}\big) - \phi v_t^{(2)} + \frac{p_0}{\gamma_1 c_0^2 R_0} p_y = 0. \tag{2.14}$$

Equations (2.6c), (2.12)–(2.14) give sufficient relation to determine the field quantities $v^{(1)}$, $v^{(2)}$, ϕ and p completely.

§ 3. Non-linear wave

We employ the reductive perturbation method [47,48] to derive a non-linear evolution equation for the shock waves propagation of the bubbly Kelvin–Voigt liquid flow. The space and time in equations (2.6c), (2.12)–(2.14) are rescaled in order to introduce space and time variables, which are appropriate for the description of long wavelength behavior. Assuming $k_x^2 \ll 1$, $k_x^2 = \varepsilon\eta^2$ and using (2.11), we have

$$k_x x - wt = \eta\left[\varepsilon^{\frac{1}{2}}(x - t) + \frac{a_{13}}{2a_{11}}\varepsilon^{\frac{3}{2}}\eta^2 t\right],$$

thus, the independent variables are stretched as

$$\xi = \varepsilon^{\frac{1}{2}}(x - t), \quad \delta = \varepsilon\kappa y, \quad \tau = \varepsilon^{\frac{3}{2}} t. \tag{3.1}$$

In (3.1), κ is a free parameter. The small parameter $\varepsilon = \frac{R_0}{l}$ measures the weakness of non-linearity; δ represents the perturbation of the wave in y direction. Substituting (3.1) into (2.6c) and (2.12)–(2.14), and equating $\varepsilon^{\frac{1}{2}}$ in (2.12)–(2.14), we get

$$\begin{aligned} &\gamma_1(\varepsilon\phi_\tau - \phi_\xi) + \gamma_1\phi\Big(v_\xi^{(1)} + \varepsilon^{\frac{1}{2}}\kappa v_\delta^{(2)}\Big) + \gamma_1\big(v^{(1)}\phi_\xi + \varepsilon^{\frac{1}{2}}\kappa v^{(2)}\phi_\delta\big) \\ &\quad - 2\gamma_2 R_0\phi(\varepsilon\phi_\tau - \phi_\xi) - \frac{\rho_0}{R_0}\Big(v_\xi^{(1)} + \varepsilon^{\frac{1}{2}}\kappa v_\delta^{(2)}\Big) = 0, \end{aligned} \tag{3.2}$$

$$\frac{\rho_0}{\gamma_1 R_0}\left(\varepsilon v^{(1)}_\tau - v^{(1)}_\xi + v^{(1)} v^{(1)}_\xi + \varepsilon^{\frac{1}{2}}\kappa v^{(2)} v^{(1)}_\delta\right) - \phi\left(\varepsilon v^{(1)}_\tau - v^{(1)}_\xi\right) + \frac{p_\xi}{a_{11}} = 0, \tag{3.3}$$

$$\frac{\rho_0}{\gamma_1 R_0}\left(\varepsilon v^{(2)}_\tau - v^{(2)}_\xi + v^{(1)} v^{(2)}_\xi + \varepsilon^{\frac{1}{2}}\kappa v^{(2)} v^{(2)}_\delta\right) - \phi\left(\varepsilon v^{(2)}_\tau - v^{(2)}_\xi\right) + \varepsilon^{\frac{1}{2}}\frac{\kappa p_\delta}{a_{11}} = 0, \tag{3.4}$$

$$\begin{aligned} p = &- a_{11}\phi - a_{12}\varepsilon^{\frac{3}{2}}\phi_\tau + a_{12}\varepsilon^{\frac{1}{2}}\phi_\xi - a_{13}\varepsilon^3\phi_{\tau\tau} + 2a_{13}\varepsilon^2\phi_{\tau\xi} \\ &- a_{13}\varepsilon\phi_{\xi\xi} + a_{14}\varepsilon^{\frac{9}{2}}\phi_{\tau\tau\tau} - 3\varepsilon^{\frac{7}{2}}a_{14}\phi_{\tau\tau\xi} + 3\varepsilon^{\frac{5}{2}}a_{14}\phi_{\tau\xi\xi} \\ &- \varepsilon^{\frac{3}{2}}a_{14}\phi_{\xi\xi\xi} + \varepsilon^{\frac{3}{2}}a_{12}\phi\phi_\tau - \varepsilon^{\frac{1}{2}}a_{12}\phi\phi_\xi - \phi_\tau^2 a_{15}\varepsilon^3 + 2\phi_\xi\varepsilon^2 a_{15}\phi_\tau \\ &- \phi_\xi^2\varepsilon a_{15} - \varepsilon^3 a_{16}\phi\phi_{\tau\tau} + 2\varepsilon^2 a_{16}\phi\phi_{\tau\xi} - \varepsilon a_{16}\phi\phi_{\xi\xi} + \varepsilon^{\frac{9}{2}}a_{17}\phi_\tau\phi_{\tau\tau} \\ &- 2\varepsilon^{\frac{7}{2}}a_{17}\phi_\tau\phi_{\tau\xi} - \varepsilon^{\frac{7}{2}}a_{17}\phi_\xi\phi_{\tau\tau} + \varepsilon^{\frac{5}{2}}a_{17}\phi_\tau\phi_{\xi\xi} + 2\varepsilon^{\frac{5}{2}}a_{17}\phi_\xi\phi_{\tau\xi} \\ &- \varepsilon^{\frac{3}{2}}a_{17}\phi_\xi\phi_{\xi\xi} + \varepsilon^{\frac{9}{2}}a_{18}\phi\phi_{\tau\tau\tau} - 3\varepsilon^{\frac{7}{2}}a_{18}\phi\phi_{\tau\tau\xi} + 3\varepsilon^{\frac{5}{2}}a_{18}\phi\phi_{\tau\xi\xi} \\ &- \varepsilon^{\frac{3}{2}}a_{18}\phi\phi_{\xi\xi\xi} + a_{19}\phi^2. \end{aligned} \tag{3.5}$$

Let the asymptotic series in ε of the field variables be

$$\begin{aligned} p &= \varepsilon p_1 + \varepsilon^2 p_2 + \dots, & \phi &= \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots, \\ v^{(1)} &= \varepsilon v^{(1)}_1 + \varepsilon^2 v^{(1)}_2 + \dots, & v^{(2)} &= \varepsilon^{\frac{3}{2}} v^{(2)}_1 + \varepsilon^{\frac{5}{2}} v^{(2)}_2 + \dots. \end{aligned} \tag{3.6}$$

Using (3.6) in (3.2)–(3.5), the following differential equations are obtained as $O(\varepsilon)$ equations:

$$\phi_{1_\xi} + \frac{\rho_0}{\gamma_1 R_0} v^{(1)}_{1_\xi} = 0, \quad -\frac{\rho_0}{\gamma_1 R_0} v^{(1)}_{1_\xi} + \frac{1}{a_{11}} p_{1_\xi} = 0, \tag{3.7a}$$

$$p_1(\xi,\delta,\tau) = -a_{11}\phi_1(\xi,\delta,\tau). \tag{3.7b}$$

Solving (3.7a), we have

$$\phi_1(\xi,\delta,\tau) = -\frac{\rho_0}{\gamma_1 R_0} v^{(1)}_1(\xi,\delta,\tau) + v^{(1)}_1(\delta,\tau), \tag{3.8}$$

where $v^{(1)}_1(\delta,\tau)$ is an arbitrary function. Assuming $v^{(1)}_1(\delta,\tau) = 0$ due to the negligible propagation of the wave in δ direction, (3.8) becomes

$$\phi_1(\xi,\delta,\tau) = -\frac{\rho_0}{\gamma_1 R_0} v^{(1)}_1(\xi,\delta,\tau).$$

Substituting (3.6) into (3.2)–(3.5) and equating at ε^2 in (3.2), (3.3), (3.5) and at $\varepsilon^{3/2}$ in (3.4), we obtain the following set of differential equations in the case $a_{12} = O(\varepsilon^{1/2})$ as

$$\phi_{1_\tau} - \phi_{2_\xi} + \phi_1 {v^{(1)}_1}_\xi + v^{(1)}_1\phi_{1_\xi} + \frac{2\gamma_2 R_0}{\gamma_1}\phi_1\phi_{1_\xi} - \frac{\rho_0}{\gamma_1 R_0}\left(v^{(1)}_{2_\xi} + \kappa v^{(2)}_{1_\delta}\right) = 0, \tag{3.9}$$

$$\frac{\rho_0}{\gamma_1 R_0}\left(v^{(1)}_{1\tau} - v^{(1)}_{2\xi} + v^{(1)}_1 v^{(1)}_{1\xi}\right) + \phi_1 v^{(1)}_{1\xi} + \frac{1}{a_{11}} p_{2\xi} = 0, \tag{3.10}$$

$$-\frac{\rho_0}{\gamma_1 R_0} v^{(2)}_{1\xi} + \frac{\kappa}{a_{11}} p_{1\delta} = 0,$$

$$p_2 = -a_{11}\phi_2 + a_{12}\phi_{1\xi} - a_{13}\phi_{1\xi\xi} + a_{19}\phi_1^2. \tag{3.11}$$

From (3.11), substituting p_2 into (3.10) and out of which $\phi_{2\xi}$ is substituted into (3.9), we have

$$\phi_{1_\tau} + \left(\frac{\gamma_2 R_0}{\gamma_1} - \frac{\gamma_1 R_0}{\rho_0} - \frac{a_{19}}{a_{11}}\right)\phi_1\phi_{1_\xi} - \frac{a_{12}}{2a_{11}}\phi_{1\xi\xi} + \frac{a_{13}}{2a_{11}}\phi_{1\xi\xi\xi} - \frac{\rho_0}{2\gamma_1 R_0}\kappa v^{(2)}_{1\delta} = 0, \tag{3.12}$$

$$-\frac{\rho_0}{\gamma_1 R_0} v^{(2)}_{1\xi\delta} - \kappa\phi_{1\delta\delta} = 0. \tag{3.13}$$

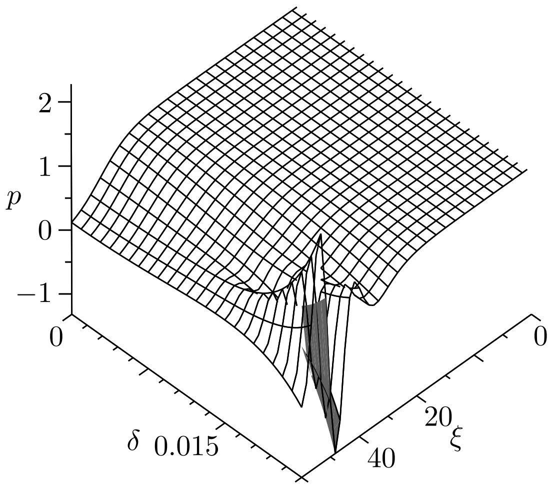

Fig. 1. Two dimensional pressure wave propagation in Kelvin–Voigt liquid for ideal gas at $\tau = 10\,\text{sec}$

Differentiating (3.12) with respect to ξ and using (3.13), we obtain

$$\left(\phi_{1_\tau} + \left(\frac{\gamma_2 R_0}{\gamma_1} - \frac{\gamma_1 R_0}{\rho_0} - \frac{a_{19}}{a_{11}}\right)\phi_1\phi_{1_\xi} - \frac{a_{12}}{2a_{11}}\phi_{1_{\xi\xi}} + \frac{a_{13}}{2a_{11}}\phi_{1_{\xi\xi\xi}}\right)_\xi + \frac{\kappa^2}{2}\phi_{1_{\delta\delta}} = 0. \tag{3.14}$$

Using (3.7b), equation (3.14) can be written in terms of pressure of the liquid in bubbly Kelvin–Voigt liquid mixture flow as

$$\left(p_{1_\tau} + q_1 p_1 p_{1_\xi} - q_2 p_{1_{\xi\xi}} + q_3 p_{1_{\xi\xi\xi}}\right)_\xi + q_4 p_{1_{\delta\delta}} = 0, \tag{3.15}$$

where q_1, q_2 and q_3 are the coefficients of the non-linearity, dissipative and dispersion terms respectively, while q_4 denotes the coefficient of the transverse propagation in δ direction, given as

$$q_1 = \left(\frac{\gamma_1 R_0}{\rho_0} + \frac{a_{19}}{a_{11}} - \frac{\gamma_2 R_0}{\gamma_1}\right)\frac{1}{a_{11}}, \quad q_2 = \frac{a_{12}}{2a_{11}}, \quad q_3 = \frac{a_{13}}{2a_{11}}, \quad q_4 = \frac{\kappa^2}{2}. \tag{3.16}$$

Equations (3.14) and (3.15) are two-dimensional Korteweg–de Vries–Burgers equations in terms of bubble radius and pressure perturbations. The near field equation (2.2) and far field equations (3.14) explain the behavior of the first-order perturbation ϕ in temporal and spatial dimensions of $O(1)$ and $O(\epsilon^2)$ respectively. It is obvious from (3.16), (2.7) and (1.5), that the shock wave propagation arising as a result of the mixture behaves according to q_1, q_2, q_3. q_1 determines the strength of the shock which depends on the R_0, p_0, σ, G, n, N, ρ_0 and volume of the cluster. The dissipation of the shock wave is mainly attributed to the effect of the viscosity of the liquid while the dispersion of the propagating shock wave is caused by the effect of modulus of elasticity.

§ 4. Result and discussion

Solutions to (2+1)-KdVB equation (3.15) are derived in [49, 50]. The analysis on the nature of shock wave is done using the initial condition [51, 52]

$$p_0(\xi, 0, \tau) = 0.5\left(1 - \tanh\frac{|\xi| - 25 - \tau}{5}\right).$$

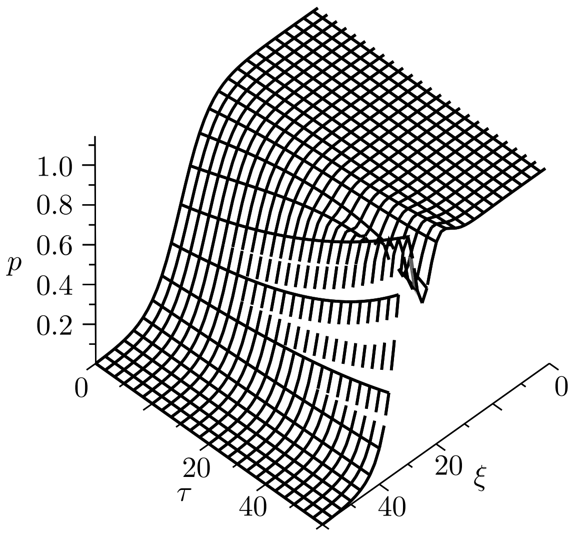

Fig. 2. Two dimensional pressure wave propagation in Kelvin–Voigt liquid for an ideal gas at $\tau = 0, \ldots, 60\,\text{sec}$

The pressure wave profile of the shock wave propagation in a weakly compressible Kelvin–Voigt liquid with bubble–bubble interaction will be analysed here. This will be done using equation (3.15). Fig. 1 shows the two dimensional shock wave propagation of the bubbly-liquid in ξ and δ directions at $\tau = 10\,\text{sec}$. Fig. 2 shows the two dimensional shock wave propagation in ξ over the range of a specified τ which shows the behavior of shock wave over time. The figure describes wave propagation from $\tau = 0, \ldots, 40\,\text{sec}$, where the oscillation is fully developed. Fig. 3 shows the two dimensional shock wave propagation in ξ over the range of $\tau = 0, \ldots, 25\,\text{sec}$ to show the start-up of the oscillatory wave. For more detailed analysis, we shall limit ourselves to the one dimensional case; the ξ direction only.

The dynamics of shock waves propagation in mono-dispersed, weakly compressible bubbly Kelvin–Voigt liquid flow with bubble–bubble interaction are investigated using the following parameters [42]: $p_0 = 101000\,\text{Pa}$, $\rho_l = 1000\,\text{kg/m}^3$, $\rho_g = 0.001\,\text{kg/m}^3$, $G = 0.5\,\text{MPa}^2$, $R_0 = 1\,\text{mm}$, $\sigma = 53.5\,\text{mN/m}$, $\mu = 140\,\text{mPa.s}$, $c_l = 1500\,\text{ms}^{-1}$ and $N = 200$, $n = 1.4$ and $K = 300 * R_0$.

Fig. 4 shows the influence of the bubble radius to pressure profile of shock wave propagation in bubbly compressible Kelvin–Voigt liquid. The figure indicates that the size of the bubble has no influence on the amplitude of the shock wave as at $\tau = 0$ up to 40 sec. Fig. 5 analyses the effect of the polytropic index which describes the thermal behavior of gas inside the bubbles. The effect of the thermal variation of the polytropic index has influence on the wave propagation of the pressure profile, the higher the polytropic index, the lower the amplitude of the shock wave. This implies that, just as in the case of velocity profile, the isothermal process will give rise to a shock wave propagation with highest amplitude. It is observed that there is a significant effect of module of elasticity on the shock wave. The more elastic the liquid, the lower the amplitude of the shock wave. Fig. 6 shows the effect of pressure variation on the pressure profile of the shock wave propagation. It is observed that change in pressure has a significant effect on the shock wave amplitude. The effect is just the higher the pressure, the higher the shock wave amplitude, and the more steeper the wave. The number of bubbles and cluster size has no effect using the parameters under consideration, this result is in agreement with the results and analysis in [25].

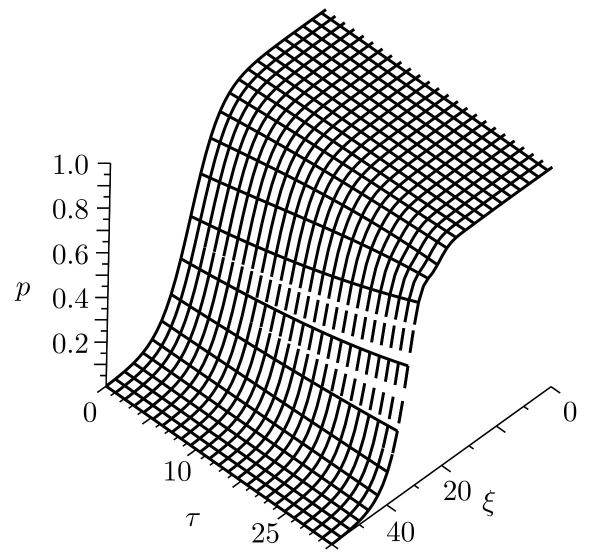

Fig. 3. Two dimensional pressure wave propagation in Kelvin–Voigt liquid for an ideal gas at $\tau = 0, \ldots, 35$ sec

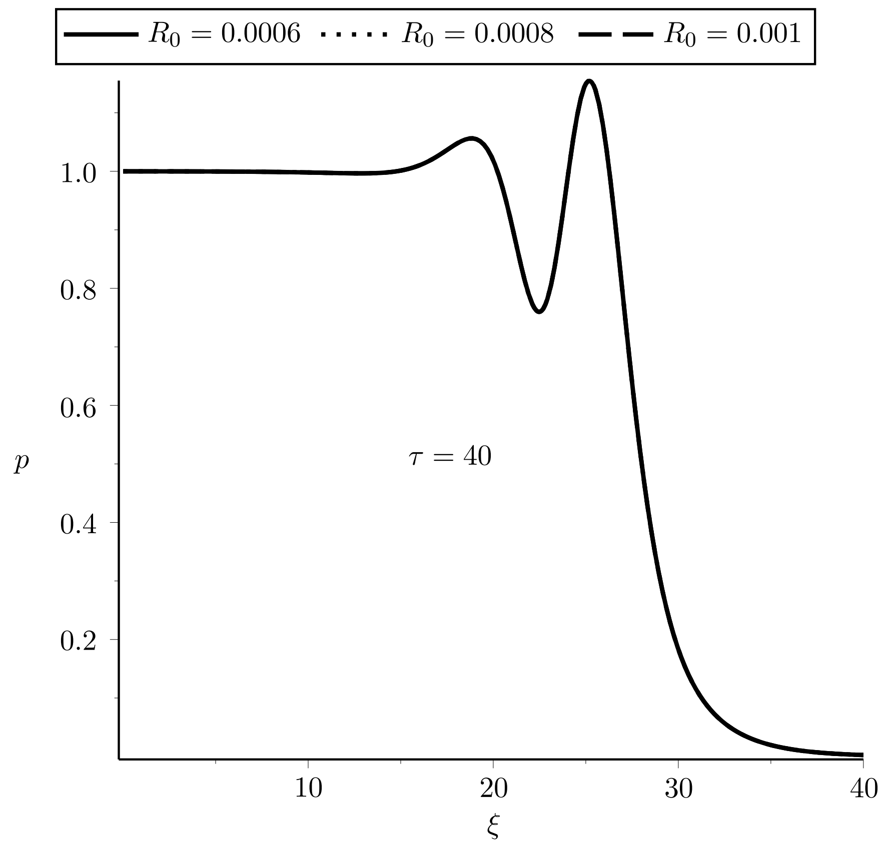

Fig. 4. Shock wave propagation in terms of pressure in Kelvin–Voigt liquid with variation of initial bubble radius

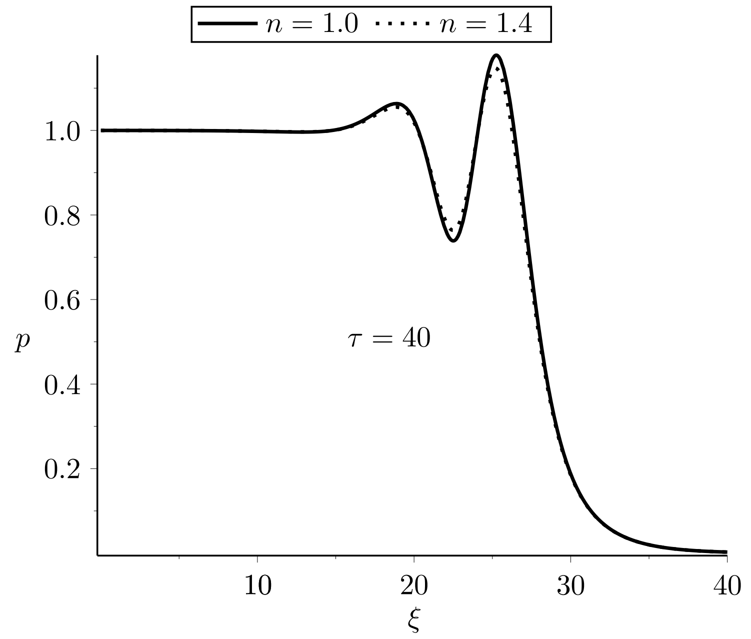

Fig. 5. Shock wave propagation in terms of pressure in Kelvin–Voigt liquid with variation of polytrophic index

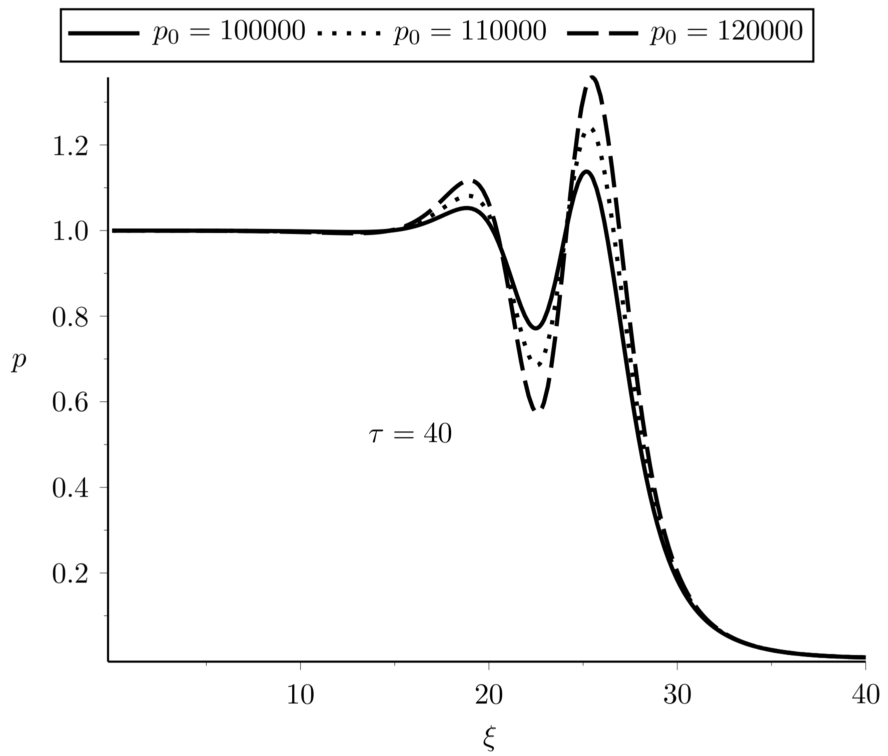

Fig. 6. Shock wave propagation in terms of pressure in Kelvin–Voigt liquid with variation of initial pressure

§ 5. Conclusion

In this paper, a modified Rayleigh–Plesset equation that describes the bubble dynamics in a weakly compressible viscoelastic liquid with due account for bubble–bubble interaction is derived using the energy conservation approach. The equation is coupled with equations of state for gas and mixture to study the shock wave propagation in bubbly Kelvin–Voigt liquid flow. The reductive perturbation method is adopted. A (2+1)-KdVB equation in terms of pressure profile is derived. An existing solution is used to graphically represent the result. Our theoretical analysis indicates that the cluster radius, interaction between bubble–bubble and the number of bubbles in a cluster have no effect on the shock wave propagation in compressible Kelvin–Voigt liquid flow taking into account the bubble cluster, whereas the initial bubble radius and pressure, viscosity, thermal property of the gas affect the shock wave in bubble viscoelastic liquid flow.

REFERENCES

1. Grandjean H., Jacques N., Zaleski S. Shock propagation in liquids containing bubbly clusters: a continuum approach, *Journal of Fluid Mechanics*, 2012, vol. 701, pp. 304–332. https://doi.org/10.1017/jfm.2012.159
2. Naseri H., Koukouvinis P., Malgarinos I., Gavaises M. On viscoelastic cavitating flows: A numerical study, *Physics of Fluids*, 2018, vol. 30, issue 3, 033102. https://doi.org/10.1063/1.5011978
3. Kudryashov N. A., Sinelshchikov D. I. Equation for the three-dimensional nonlinear waves in liquid with gas bubbles, *Physica Scripta*, 2012, vol. 85, no. 2, 025402. https://doi.org/10.1088/0031-8949/85/02/025402
4. Brennen C. E. *Cavitation and bubble dynamics*, Cambridge: Cambridge University Press, 2013. https://doi.org/10.1017/CBO9781107338760
5. Tukhvatullina R. R., Frolov S. M. Numerical simulation of shock and detonation waves in bubbly liquids, *Shock Waves*, 2020, vol. 30, issue 3, pp. 263–271. https://doi.org/10.1007/s00193-019-00914-2
6. Brujan E. A. The equation of bubble dynamics in a compressible linear viscoelastic liquid, *Fluid Dynamics Research*, 2001, vol. 29, issue 5, pp. 287–294. https://doi.org/10.1016/S0169-5983(01)00030-2
7. Hua C., Johnsen E. Nonlinear oscillations following the Rayleigh collapse of a gas bubble in a linear viscoelastic (tissue-like) medium, *Physics of Fluids*, 2013, vol. 25, issue 8, 083101. https://doi.org/10.1063/1.4817673
8. Brujan E. *Cavitation in non-Newtonian fluids*, Berlin, Heidelberg: Springer, 2011. https://doi.org/10.1007/978-3-642-15343-3
9. Fuster D., Dopazo C., Hauke G. Liquid compressibility effects during the collapse of a single cavitating bubble, *The Journal of the Acoustical Society of America*, 2011, vol. 129, issue 1, pp. 122–131. https://doi.org/10.1121/1.3502464
10. Warnez M. T., Johnsen E. Numerical modeling of bubble dynamics in viscoelastic media with relaxation, *Physics of Fluids*, 2015, vol. 27, issue 6, 063103. https://doi.org/10.1063/1.4922598
11. Taklifi A., Hanafizadeh P., Aliabadi A., Akhavan-Behabadi M. A. Nonlinear radial oscillations of micro-bubbles in a compressible UCM medium, *Particulate Science and Technology*, 2016, vol. 35, issue 1, pp. 86–92. https://doi.org/10.1080/02726351.2015.1131790
12. Lang C., Boolakee O., Schmidt S. J., Adams N. A. A compressible 3D finite volume approach for the simulation of unsteady viscoelastic cavitating flows, *International Journal of Multiphase Flow*, 2022, vol. 150, 103981. https://doi.org/10.1016/j.ijmultiphaseflow.2022.103981
13. Ando K., Colonius T., Brennen C. E. Numerical simulation of shock propagation in a polydisperse bubbly liquid, *International Journal of Multiphase Flow*, 2011, vol. 37, issue 6, pp. 596–608. https://doi.org/10.1016/j.ijmultiphaseflow.2011.03.007
14. Ando K., Colonius T., Brennen C. E. Shock propagation in polydisperse bubbly liquids, *Bubble dynamics and shock waves*, Berlin, Heidelberg: Springer, 2013, pp. 141–175. https://doi.org/10.1007/978-3-642-34297-4_5

15. Seo J. H., Lele S. K., Tryggvason G. Investigation and modeling of bubble-bubble interaction effect in homogeneous bubbly flows, *Physics of Fluids*, 2010, vol. 22, issue 6, 063302. https://doi.org/10.1063/1.3432503
16. Gimaltdinov I. K., Lepikhin S. A. Study of postdetonation waves after the counter collision of detonation waves in a bubble liquid, *High Temperature*, 2021, vol. 59, issues 2–6, pp. 277–282. https://doi.org/10.1134/S0018151X21020036
17. Kubota A., Kato H., Yamaguchi H. A new modelling of cavitating flows: a numerical study of unsteady cavitation on a hydrofoil section, *Journal of Fluid Mechanics*, 1992, vol. 240, pp. 59–96. https://doi.org/10.1017/S002211209200003X
18. Chahine G. L., Duraiswami R. Dynamical interactions in a multi-bubble cloud, *Journal of Fluids Engineering*, 1992, vol. 114, issue 4, pp. 680–686. https://doi.org/10.1115/1.2910085
19. Chahine G. L. Cloud cavitation: theory, *14th Symposium on Naval Hydrodynamics*, Washington (DC): National Academy Press, 1983, pp. 165–194.
20. Delale C. F., Nas S., Tryggvason G. Direct numerical simulations of shock propagation in bubbly liquids, *Physics of Fluids*, 2005, vol. 17, issue 12, 121705. https://doi.org/10.1063/1.2158431
21. Delale C. F., Tryggvason G. Shock structure in bubbly liquids: comparison of direct numerical simulations and model equations, *Shock Waves*, 2008, vol. 17, issue 6, pp. 433–440. https://doi.org/10.1007/s00193-008-0126-1
22. Fuster D. A review of models for bubble clusters in cavitating flows, *Flow, Turbulence and Combustion*, 2019, vol. 102, issue 3, pp. 497–536. https://doi.org/10.1007/s10494-018-9993-4
23. Stavropoulos–Vasilakis E., Kyriazis N., Jadidbonab H., Koukouvinis P., Gavaises M. Review of numerical methodologies for modeling cavitation, *Cavitation and bubble dynamics*, Academic Press, 2021, pp. 1–35. https://doi.org/10.1016/B978-0-12-823388-7.00004-7
24. Gubaidullin A. A., Beregova O. Sh., Bekishev S. A. Shock waves in non-Newtonian bubbly liquids, *International Journal of Multiphase Flow*, 2001, vol. 27, issue 4, pp. 635–655. https://doi.org/10.1016/S0301-9322(00)00030-6
25. Fuster D., Colonius T. Modelling bubble clusters in compressible liquids, *Journal of Fluid Mechanics*, 2011, vol. 688, pp. 352–389. https://doi.org/10.1017/jfm.2011.380
26. Bunner B., Tryggvason G. Dynamics of homogeneous bubbly flows. Part 1. Rise velocity and microstructure of the bubbles, *Journal of Fluid Mechanics*, 2002, vol. 466, pp. 17–52. https://doi.org/10.1017/S0022112002001179
27. Bunner B., Tryggvason G. Effect of bubble deformation on the properties of bubbly flows, *Journal of Fluid Mechanics*, 2003, vol. 495, pp. 77–118. https://doi.org/10.1017/S0022112003006293
28. Colonius T., d'Auria F., Brennen C. E. Acoustic saturation in bubbly cavitating flow adjacent to an oscillating wall, *Physics of Fluids*, 2000, vol. 12, issue 11, pp. 2752–2761. https://doi.org/10.1063/1.1313561
29. Preston A. T., Colonius T., Brennen C. E. A numerical investigation of unsteady bubbly cavitating nozzle flows, *Physics of Fluids*, 2002, vol. 14, issue 1, pp. 300–311. https://doi.org/10.1063/1.1416497
30. Wang Yi-Chun, Brennen C. E. Numerical computation of shock waves in a spherical cloud of cavitation bubbles, *Journal of Fluids Engineering*, 1999, vol. 121, issue 4, pp. 872–880. https://doi.org/10.1115/1.2823549
31. Nigmatulin R. I. Mathematical modelling of bubbly liquid motion and hydrodynamical effects in wave propagation phenomenon, *Mechanics and physics of bubbles in liquids. Proceedings IUTAM Symposium, held in Pasadena, California, 15–19 June 1981*, Dordrecht: Springer, 1982, pp. 267–289. https://doi.org/10.1007/978-94-009-7532-3_25
32. Nakoryakov V. E., Pokusaev B. G., Shreiber I. R. *Wave propagation in gas–liquid media*, CRC Press, 1993.
33. Prosperetti A. The equation of bubble dynamics in a compressible liquid, *The Physics of Fluids*, 1987, vol. 30, issue 11, pp. 3626–3628. https://doi.org/10.1063/1.866445
34. Doinikov A. A. Equations of coupled radial and translational motions of a bubble in a weakly compressible liquid, *Physics of Fluids*, 2005, vol. 17, issue 12, 128101. https://doi.org/10.1063/1.2145430

35. Prosperetti A. Bubble phenomena in sound fields: part one, *Ultrasonics*, 1984, vol. 22, issue 2, pp. 69–77. https://doi.org/10.1016/0041-624X(84)90024-6
36. Kafiabad H. A., Sadeghy K. Chaotic behavior of a single spherical gas bubble surrounded by a Giesekus liquid: A numerical study, *Journal of Non-Newtonian Fluid Mechanics*, 2010, vol. 165, issues 13–14, pp. 800–811. https://doi.org/10.1016/j.jnnfm.2010.04.010
37. Kim C. Collapse of spherical bubbles in Maxwell fluids, *Journal of Non-Newtonian Fluid Mechanics*, 1994, vol. 55, issue 1, pp. 37–58. https://doi.org/10.1016/0377-0257(94)80059-6
38. Delale C. F., Schnerr G. H., Sauer J. Quasi-one-dimensional steady-state cavitating nozzle flows, *Journal of Fluid Mechanics*, 2001, vol. 427, pp. 167–204. https://doi.org/10.1017/S0022112000002330
39. Ilinskii Yu. A., Zabolotskaya E. A. Cooperative radiation and scattering of acoustic waves by gas bubbles in liquids, *The Journal of the Acoustical Society of America*, 1992, vol. 92, issue 5, pp. 2837–2841. https://doi.org/10.1121/1.404403
40. Oskolkov A. P. Theory of nonstationary flows of Kelvin–Voigt fluids, *Journal of Soviet Mathematics*, 1985, vol. 28, issue 5, pp. 751–758. https://doi.org/10.1007/BF02112340
41. Albernaz D. L., Cunha F. R. Bubble dynamics in a Maxwell fluid with extensional viscosity, *Mechanics Research Communications*, 2011, vol. 38, issue 3, pp. 255–260. https://doi.org/10.1016/j.mechrescom.2011.02.013
42. Yang X., Church C. C. Nonlinear dynamics of gas bubbles in viscoelastic media, *Acoustics Research Letters Online*, 2005, vol. 6, issue 3, pp. 151–156. https://doi.org/10.1121/1.1897824
43. Kundu S., Pani A. K. Stabilization of Kelvin–Voigt viscoelastic fluid flow model, *Applicable Analysis*, 2019, vol. 98, issue 12, pp. 2284–2307. https://doi.org/10.1080/00036811.2018.1460810
44. Wazwaz A.-M. *Partial differential equations and solitary waves theory*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-00251-9
45. Prosperetti A. Bubble dynamics: a review and some recent results, *Applied Scientific Research*, 1982, vol. 38, issue 1, pp. 145–164. https://doi.org/10.1007/BF00385945
46. van Wijngaarden L. On the equations of motion for mixtures of liquid and gas bubbles, *Journal of Fluid Mechanics*, 1968, vol. 33, issue 3, pp. 465–474. https://doi.org/10.1017/S002211206800145X
47. Leblond H. The reductive perturbation method and some of its applications, *Journal of Physics B: Atomic, Molecular and Optical Physics*, 2008, vol. 41, issue 4, 043001. https://doi.org/10.1088/0953-4075/41/4/043001
48. Shah A., Saeed R. Ion acoustic shock waves in a relativistic electron–positron–ion plasmas, *Physics Letters A*, 2009, vol. 373, issue 45, pp. 4164–4168. https://doi.org/10.1016/j.physleta.2009.09.028
49. Kaya D. An application of the decomposition method for the two-dimensional KdV–Burgers equation, *Computers and Mathematics with Applications*, 2004, vol. 48, issues 10–11, pp. 1659–1665. https://doi.org/10.1016/j.camwa.2004.02.008
50. Zabihi F., Saffarian M. A meshless method using radial basis functions for numerical solution of the two-dimensional KdV–Burgers equation, *The European Physical Journal Plus*, 2016, vol. 131, issue 7, article number: 243. https://doi.org/10.1140/epjp/i2016-16243-y
51. Yan Xu, Chi-wang Shu. Local discontinuous Galerkin methods for three classes of nonlinear wave equations, *Journal of Computational Mathematics*, 2004, vol. 22, issue 2, pp. 250–274. https://global-sci.org/intro/article_detail/jcm/10327.html
52. Zaki S. I. Solitary waves of the Korteweg–de Vries–Burgers' equation, *Computer Physics Communications*, 2000, vol. 126, issue 3, pp. 207–218. https://doi.org/10.1016/S0010-4655(99)00470-1

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Распространение нелинейных волн в слабосжимаемой жидкости Кельвина–Фойгта, содержащей пузырьковые кластеры

Ключевые слова: ударная волна, жидкость Кельвина–Фойгта, пузырьковая жидкость, уравнение Кортевега–де Фриза–Бюргера.

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С помощью упрощенного метода возмущений исследуется влияние взаимодействия между пузырьками на распространение волн в однородном слабосжимаемом вязкоупругом пузырьковом потоке. С использованием подхода сохранения кинетической энергии выводится уравнение динамики пузырьков. Динамика пузырьков и уравнения смеси в сочетании с уравнением состояния газа позволяют исследовать явление распространения ударной волны в смеси. Выведено двумерное уравнение Кортевега–де Фриза–Бюргера в терминах профиля давления. Установлено, что при использовании рассматриваемых нами параметров взаимодействие между пузырьками не оказывает влияния.

СПИСОК ЛИТЕРАТУРЫ

1. Grandjean H., Jacques N., Zaleski S. Shock propagation in liquids containing bubbly clusters: a continuum approach // Journal of Fluid Mechanics. 2012. Vol. 701. P. 304–332. https://doi.org/10.1017/jfm.2012.159
2. Naseri H., Koukouvinis P., Malgarinos I., Gavaises M. On viscoelastic cavitating flows: A numerical study // Physics of Fluids. 2018. Vol. 30. Issue 3. 033102. https://doi.org/10.1063/1.5011978
3. Kudryashov N. A., Sinelshchikov D. I. Equation for the three-dimensional nonlinear waves in liquid with gas bubbles // Physica Scripta. 2012. Vol. 85. No. 2. 025402. https://doi.org/10.1088/0031-8949/85/02/025402
4. Brennen C. E. Cavitation and bubble dynamics. Cambridge: Cambridge University Press, 2013. https://doi.org/10.1017/CBO9781107338760
5. Tukhvatullina R. R., Frolov S. M. Numerical simulation of shock and detonation waves in bubbly liquids // Shock Waves. 2020. Vol. 30. Issue 3. P. 263–271. https://doi.org/10.1007/s00193-019-00914-2
6. Brujan E. A. The equation of bubble dynamics in a compressible linear viscoelastic liquid // Fluid Dynamics Research. 2001. Vol. 29. Issue 5. P. 287–294. https://doi.org/10.1016/S0169-5983(01)00030-2
7. Hua C., Johnsen E. Nonlinear oscillations following the Rayleigh collapse of a gas bubble in a linear viscoelastic (tissue-like) medium // Physics of Fluids. 2013. Vol. 25. Issue 8. 083101. https://doi.org/10.1063/1.4817673
8. Brujan E. Cavitation in non-Newtonian fluids. Berlin, Heidelberg: Springer, 2011. https://doi.org/10.1007/978-3-642-15343-3
9. Fuster D., Dopazo C., Hauke G. Liquid compressibility effects during the collapse of a single cavitating bubble // The Journal of the Acoustical Society of America. 2011. Vol. 129. Issue 1. P. 122–131. https://doi.org/10.1121/1.3502464
10. Warnez M. T., Johnsen E. Numerical modeling of bubble dynamics in viscoelastic media with relaxation // Physics of Fluids. 2015. Vol. 27. Issue 6. 063103. https://doi.org/10.1063/1.4922598
11. Taklifi A., Hanafizadeh P., Aliabadi A., Akhavan-Behabadi M. A. Nonlinear radial oscillations of micro-bubbles in a compressible UCM medium // Particulate Science and Technology. 2016. Vol. 35. Issue 1. P. 86–92. https://doi.org/10.1080/02726351.2015.1131790
12. Lang C., Boolakee O., Schmidt S. J., Adams N. A. A compressible 3D finite volume approach for the simulation of unsteady viscoelastic cavitating flows // International Journal of Multiphase Flow. 2022. Vol. 150. 103981. https://doi.org/10.1016/j.ijmultiphaseflow.2022.103981
13. Ando K., Colonius T., Brennen C. E. Numerical simulation of shock propagation in a polydisperse bubbly liquid // International Journal of Multiphase Flow. 2011. Vol. 37. Issue 6. P. 596–608. https://doi.org/10.1016/j.ijmultiphaseflow.2011.03.007

14. Ando K., Colonius T., Brennen C. E. Shock propagation in polydisperse bubbly liquids // Bubble dynamics and shock waves. Berlin, Heidelberg: Springer, 2013. P. 141–175. https://doi.org/10.1007/978-3-642-34297-4_5
15. Seo J. H., Lele S. K., Tryggvason G. Investigation and modeling of bubble-bubble interaction effect in homogeneous bubbly flows // Physics of Fluids. 2010. Vol. 22. Issue 6. 063302. https://doi.org/10.1063/1.3432503
16. Gimaltdinov I. K., Lepikhin S. A. Study of postdetonation waves after the counter collision of detonation waves in a bubble liquid // High Temperature. 2021. Vol. 59. Issues 2–6. P. 277–282. https://doi.org/10.1134/S0018151X21020036
17. Kubota A., Kato H., Yamaguchi H. A new modelling of cavitating flows: a numerical study of unsteady cavitation on a hydrofoil section // Journal of Fluid Mechanics. 1992. Vol. 240. P. 59–96. https://doi.org/10.1017/S002211209200003X
18. Chahine G. L., Duraiswami R. Dynamical interactions in a multi-bubble cloud // Journal of Fluids Engineering. 1992. Vol. 114. Issue 4. P. 680–686. https://doi.org/10.1115/1.2910085
19. Chahine G. L. Cloud cavitation: theory // 14th Symposium on Naval Hydrodynamics. Washington (DC): National Academy Press, 1983. P. 165–194.
20. Delale C. F., Nas S., Tryggvason G. Direct numerical simulations of shock propagation in bubbly liquids // Physics of Fluids. 2005. Vol. 17. Issue 12. 121705. https://doi.org/10.1063/1.2158431
21. Delale C. F., Tryggvason G. Shock structure in bubbly liquids: comparison of direct numerical simulations and model equations // Shock Waves. 2008. Vol. 17. Issue 6. P. 433–440. https://doi.org/10.1007/s00193-008-0126-1
22. Fuster D. A review of models for bubble clusters in cavitating flows // Flow, Turbulence and Combustion. 2019. Vol. 102. Issue 3. P. 497–536. https://doi.org/10.1007/s10494-018-9993-4
23. Stavropoulos–Vasilakis E., Kyriazis N., Jadidbonab H., Koukouvinis P., Gavaises M. Review of numerical methodologies for modeling cavitation // Cavitation and bubble dynamics. Academic Press, 2021. P. 1–35. https://doi.org/10.1016/B978-0-12-823388-7.00004-7
24. Gubaidullin A. A., Beregova O. Sh., Bekishev S. A. Shock waves in non-Newtonian bubbly liquids // International Journal of Multiphase Flow. 2001. Vol. 27. Issue 4. P. 635–655. https://doi.org/10.1016/S0301-9322(00)00030-6
25. Fuster D., Colonius T. Modelling bubble clusters in compressible liquids // Journal of Fluid Mechanics. 2011. Vol. 688. P. 352–389. https://doi.org/10.1017/jfm.2011.380
26. Bunner B., Tryggvason G. Dynamics of homogeneous bubbly flows. Part 1. Rise velocity and microstructure of the bubbles // Journal of Fluid Mechanics. 2002. Vol. 466. P. 17–52. https://doi.org/10.1017/S0022112002001179
27. Bunner B., Tryggvason G. Effect of bubble deformation on the properties of bubbly flows // Journal of Fluid Mechanics. 2003. Vol. 495. P. 77–118. https://doi.org/10.1017/S0022112003006293
28. Colonius T., d'Auria F., Brennen C. E. Acoustic saturation in bubbly cavitating flow adjacent to an oscillating wall // Physics of Fluids. 2000. Vol. 12. Issue 11. P. 2752–2761. https://doi.org/10.1063/1.1313561
29. Preston A. T., Colonius T., Brennen C. E. A numerical investigation of unsteady bubbly cavitating nozzle flows // Physics of Fluids. 2002. Vol. 14. Issue 1. P. 300–311. https://doi.org/10.1063/1.1416497
30. Wang Yi-Chun, Brennen C. E. Numerical computation of shock waves in a spherical cloud of cavitation bubbles // Journal of Fluids Engineering. 1999. Vol. 121. Issue 4. P. 872–880. https://doi.org/10.1115/1.2823549
31. Nigmatulin R. I. Mathematical modelling of bubbly liquid motion and hydrodynamical effects in wave propagation phenomenon // Mechanics and physics of bubbles in liquids. Proceedings IUTAM Symposium, held in Pasadena, California, 15–19 June 1981. Dordrecht: Springer, 1982. P. 267–289. https://doi.org/10.1007/978-94-009-7532-3_25
32. Накоряков В. Е., Покусаев Б. Г., Шрейбер И. Р. Волновая динамика газо- и парожидкостных сред. М.: Энергоатомиздат, 1990.
33. Prosperetti A. The equation of bubble dynamics in a compressible liquid // The Physics of Fluids. 1987. Vol. 30. Issue 11. P. 3626–3628. https://doi.org/10.1063/1.866445

34. Doinikov A. A. Equations of coupled radial and translational motions of a bubble in a weakly compressible liquid // Physics of Fluids. 2005. Vol. 17. Issue 12. 128101. https://doi.org/10.1063/1.2145430
35. Prosperetti A. Bubble phenomena in sound fields: part one // Ultrasonics. 1984. Vol. 22. Issue 2. P. 69–77. https://doi.org/10.1016/0041-624X(84)90024-6
36. Kafiabad H. A., Sadeghy K. Chaotic behavior of a single spherical gas bubble surrounded by a Giesekus liquid: A numerical study // Journal of Non-Newtonian Fluid Mechanics. 2010. Vol. 165. Issues 13–14. P. 800–811. https://doi.org/10.1016/j.jnnfm.2010.04.010
37. Kim C. Collapse of spherical bubbles in Maxwell fluids // Journal of Non-Newtonian Fluid Mechanics. 1994. Vol. 55. Issue 1. P. 37–58. https://doi.org/10.1016/0377-0257(94)80059-6
38. Delale C. F., Schnerr G. H., Sauer J. Quasi-one-dimensional steady-state cavitating nozzle flows // Journal of Fluid Mechanics. 2001. Vol. 427. P. 167–204. https://doi.org/10.1017/S0022112000002330
39. Ilinskii Yu. A., Zabolotskaya E. A. Cooperative radiation and scattering of acoustic waves by gas bubbles in liquids // The Journal of the Acoustical Society of America. 1992. Vol. 92. Issue 5. P. 2837–2841. https://doi.org/10.1121/1.404403
40. Осколков А. П. К теории нестационарных течений жидкостей Кельвина–Фойгта // Записки научных семинаров ЛОМИ. 1982. Т. 115. С. 192–202. https://www.mathnet.ru/rus/znsl4051
41. Albernaz D. L., Cunha F. R. Bubble dynamics in a Maxwell fluid with extensional viscosity // Mechanics Research Communications. 2011. Vol. 38. Issue 3. P. 255–260. https://doi.org/10.1016/j.mechrescom.2011.02.013
42. Yang X., Church C. C. Nonlinear dynamics of gas bubbles in viscoelastic media // Acoustics Research Letters Online. 2005. Vol. 6. Issue 3. P. 151–156. https://doi.org/10.1121/1.1897824
43. Kundu S., Pani A. K. Stabilization of Kelvin–Voigt viscoelastic fluid flow model // Applicable Analysis. 2019. Vol. 98. Issue 12. P. 2284–2307. https://doi.org/10.1080/00036811.2018.1460810
44. Wazwaz A.-M. Partial differential equations and solitary waves theory. Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-00251-9
45. Prosperetti A. Bubble dynamics: a review and some recent results // Applied Scientific Research. 1982. Vol. 38. Issue 1. P. 145–164. https://doi.org/10.1007/BF00385945
46. van Wijngaarden L. On the equations of motion for mixtures of liquid and gas bubbles // Journal of Fluid Mechanics. 1968. Vol. 33. Issue 3. P. 465–474. https://doi.org/10.1017/S002211206800145X
47. Leblond H. The reductive perturbation method and some of its applications // Journal of Physics B: Atomic, Molecular and Optical Physics. 2008. Vol. 41. Issue 4. 043001. https://doi.org/10.1088/0953-4075/41/4/043001
48. Shah A., Saeed R. Ion acoustic shock waves in a relativistic electron–positron–ion plasmas // Physics Letters A. 2009. Vol. 373. Issue 45. P. 4164–4168. https://doi.org/10.1016/j.physleta.2009.09.028
49. Kaya D. An application of the decomposition method for the two-dimensional KdV–Burgers equation // Computers and Mathematics with Applications. 2004. Vol. 48. Issues 10–11. P. 1659–1665. https://doi.org/10.1016/j.camwa.2004.02.008
50. Zabihi F., Saffarian M. A meshless method using radial basis functions for numerical solution of the two-dimensional KdV–Burgers equation // The European Physical Journal Plus. 2016. Vol. 131. Issue 7. Article number: 243. https://doi.org/10.1140/epjp/i2016-16243-y
51. Yan Xu, Chi-wang Shu. Local discontinuous Galerkin methods for three classes of nonlinear wave equations // Journal of Computational Mathematics. 2004. Vol. 22. Issue 2. P. 250–274. https://global-sci.org/intro/article_detail/jcm/10327.html
52. Zaki S. I. Solitary waves of the Korteweg–de Vries–Burgers’ equation // Computer Physics Communications. 2000. Vol. 126. Issue 3. P. 207–218. https://doi.org/10.1016/S0010-4655(99)00470-1

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