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Δ-FUNCTIONS ON RECURRENT RANDOM WALKS

If a random walk on a countable infinite state space is reversible, there are known necessary and sufficient conditions for the walk to be recurrent. When the condition of reversibility is dropped, by using discrete Dirichlet solutions and balayage (concepts familiar in potential theory) one could partially retrieve some of the above results concerning the recurrence and the transience of the random walk.

Keywords: parabolic networks, Dirichlet solutions, balayage, recurrent random walks.

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Introduction

A random walk $\{N, P = [p(a,b)]\}$ where N is a countable infinite state space and $P = [p(a,b)]$ is a matrix of a transition probabilities $\{p(a,b)\}$ is recurrent if the walk starting at a state e returns to e infinitely often. If the random walk is reversible (that is, there exists a function $\phi(x) > 0$ such that $\phi(a)p(a,b) = \phi(b)p(a,b)$ for any pair of states a, b) then there are known necessary and sufficient conditions for the random walk to be recurrent, proved by using methods from normed spaces. But when the random walk is not reversible, these methods are not of use. Many problems in a random walk are solved by considering it as a reversible Markov chain. We show that the reversible condition can be ignored by using potential theoretic techniques for some random walk problems. Many authors have investigated random walks in an infinite network using the Laplace operator, recall S. McGuinness [1], V. Anandam [2, 3], K. Abodayeh, V. Anandam [4, 5], C. St. J. A. Nash–Williams [6], T. Lyons [7], W. Woess [8], J. M. Cohen et al. [9], F. Colonna, M. Tjani [10], J. M. Cohen et al. [11]. In [6], Nash–Williams explained a random walk on an electrical network with the help of probabilistic methods.

Later in [7], T. Lyons studied the Royden criterion in Riemann surfaces, giving a necessary and sufficient condition for a reversible countable state Markov chain to be transient. V. Anandam [2] studied random walks in an infinite network without reversible conditions and proved the Nash–Williams criterion by using potential theoretic methods. In [12], V. Anandam and M. Damlakhi studied these potential theoretic methods in finite networks with the help of perturbed Laplace operators. K. Abodayeh, V. Anandam in [13, 14] investigated Schrödinger networks and their Cartesian product and supersolutions of discrete Schrödinger equations. In [15], N. Nathiya, Ch. Amulya Smyrna studied the developments of infinite Schrödinger networks in the Euclidean spaces. In [16], V. Anandam studied recurrent or transient random walk on an infinite tree with the help of reversibility condition and transition probabilities matrix. Whereas in this article, we have developed the potential theoretic methods without the condition of reversibility. With the help of this condition, we have studied the potential theoretic methods on infinite random walks. For example, among other results, it is shown that the random walk $\{N, P = [p(a,b)]\}$, reversible or not, is recurrent if in the associated infinite network $\{N, p(a,b)\}$ there exists a function $v(a) \geq 0$ outside a finite set such that $(1-p)v(a) \geq 0$ and $\lim\limits_{n\to\infty} v(a) = \infty$, or if every function $s(a) \geq 0$ on N such that $(1-p)s(a) \geq 0$ is constant, by making extensive use of Dirichlet solutions and balayage.

§ 1. Infinite network

In this section, an abridged version of potential theory on infinite graphs, relevant to the study of random walks, is given. It is mainly taken from [3]. Let N be an infinite graph with a countable infinite number of vertices and countable number of edges. If a and b are two vertices joined by an edge, say that a and b are neighbours denoted by $a \sim b$. The graph is connected if for any two vertices a and b, there exists a path $\{a = a_0, a_1, \ldots, a_n = b\}$, $a_i \sim a_{i+1}$ for $0 \le i \le n-1$; if $a = b$ and $n \ge 3$ in this path, say that there is a self-loop at the vertex a; say that the graph is locally-finite if every vertex has only a finite number of neighbours. A collection of real numbers $t(a,b) \ge 0$ defines a set of transition indices $\{t(a,b)\}$ on the graph, provide that $t(a,b) > 0$ if and only if $a \sim b$, $t(a,b)$ and $t(b,a)$ need not have the same value. An infinite graph N that is connected, locally finite, without self-loops and provided with a set of transition indices $\{t(a,b)\}$ is here referred to as an infinite network $\{N, t(a,b)\}$.

Let A be a subset of N. A vertex $a \in A$ is an interior vertex of A if all the neighbours $b \sim a$ are also in A. Let $\overset{\circ}{A}$ denote the collection of all the interior vertices of A, the set $\partial A = A \setminus \overset{\circ}{A}$ is referred to as the boundary of A. If $u(a)$ is a real-valued function on A, the Laplacian at a vertex $a \in \overset{\circ}{A}$ is defined as $\Delta u(a) = \sum_{b \sim a} t(a,b)\,[u(b) - u(a)]$. A real-valued function $u(a)$ on A is said to be upper Δ-function on A if $\Delta u(a) \le 0$ at every vertex $a \in \overset{\circ}{A}$ and lower Δ-function on A if $\Delta u(a) \ge 0$ at every vertex $a \in \overset{\circ}{A}$, and Δ-function on A if it is both upper Δ-function and lower Δ-function on A. A non-negative upper Δ-function $p(a)$ on A is called a basis function if it has the following property: for any lower Δ-function $v(a)$ on A such that $v(a) \le p(a)$, one should have $v(a) \ge 0$.

§ 2. Some properties of upper Δ-functions

1. If $u_n(a)$ is a sequence of upper Δ-functions on A and if $u(a) = \lim_{n\to\infty} u_n(a)$ exists and is real-valued, then $u(a)$ is upper Δ-function on A, also $\Delta u(a) = \lim_{n\to\infty} \Delta u_n(a)$.

2. If $\{v_i(a)\}$ is the family ψ of all upper Δ-functions on A majorized by on upper Δ-function $u(a)$ on A, then the family ψ is upper-directed and $h(a) = \sup_{\psi} v_i(a)$ is a Δ-function $u(a)$ on A. It is easy to remark that $p(a) = u(a) - h(a)$ is a basis function on A. Consequently, one can assert: if $u(a)$ is an upper Δ-function on A majorizing a lower Δ-function then $u(a)$ is the sum of a basis function $p(a)$ on A and its greatest Δ-function minorant $h(a)$; this decomposition as the sum of a basis function and the greatest Δ-function minorant is also unique. This is usually referred to as the Riesz decomposition.

3. **Dirichlet solution:** Many properties (like condenser principle, balayage, reduced functions etc.) in the study of basis functions on an infinite network appear as solutions to problems which are actually variations of a generalized Dirichlet problem. We shall refer to the following result [3, Theorem 3.1.7] as a

   Generalized Dirichlet solution: Let F be a subset in the network $\{N, t(a,b)\}$ and $A \subset \overset{\circ}{F}$. Suppose $f(a)$ is a real-valued function defined on F/A such that $v \le f \le u$ on F/A where u is an upper Δ-function on F and v is a lower Δ-function on F. Then there exists a function $h(a)$ on F such that $v \le h \le u$ on F; $h(a) = f(a)$ on F/A; and $\Delta u(a) = 0$ at every vertex in $\overset{\circ}{A}$. Moreover, h can be so chosen that if h_1 is another such function on F having these three properties, then $h_1 \le h$. However if the set A contains only a finite number of vertices, then the solution $h(a)$ is uniquely determined.

4. **Reduced functions and balayage:** Suppose $s(a) \geq 0$ is an upper Δ-function on a subset A and $E \subset A$. Let ψ be the family of all the non-negative upper Δ-functions $u(a)$ on A which majorize $s(a)$ on E. Then $\left(R_s^E(x)\right)_A = \inf\limits_{u \in \psi} u(a)$ is referred to as (the reduced function in the case of potential theory on topological spaces) the balayage of $s(a)$ on E in the subset A. We leave out A when it is the whole set N.

   Note that $v(a) = \left(R_s^E(a)\right)_A$ is a non-negative upper Δ-function on A such that $v(a) \leq s(a)$ on A; $v(a) = s(a)$ on E; and $\Delta v(a) = 0$ if $a \in A/E$. If there exists a basis function $p(a)$ on A such that $s(a) \leq p(a)$ on A, then $v(a)$ is a basis function on A.

5. **Parabolic and hyperbolic networks:** In the study of lower Δ-functions, upper Δ-functions in the Euclidean case, there is a marked difference between $\mathbb{R}^2$ and $\mathbb{R}^n, n \geq 3$, because of the fact that any non-negative upper Δ-function in $\mathbb{R}^2$ is constant (recall Liouville's Theorem) while non-constant positive upper Δ-functions exist in $\mathbb{R}^n, n \geq 3$ (recall the Newtonian gravitational kernel in $\mathbb{R}^3$). However, there are many similarities also in these two cases since the potential theory is based on the logarithmic kernel $\log \frac{1}{|a-b|}$ in $\mathbb{R}^2$ while in $\mathbb{R}^3$ it is based on the Newtonian Kernel $\frac{1}{|a-b|}$.

To consider these two different cases in the context of an infinite network $\{N, t\}$, let us say that it is a parabolic network if any non-negative upper Δ-function on N is constant and it is a hyperbolic network if there are non-constant positive upper Δ-functions (and hence positive basis functions) on N. There are various distinguishing properties to differentiate between these two types of networks. One such is given now by using the Dirichlet solution.

Let e be a fixed vertex in N. Let $\{A_n\}$ be a sequence of finite sets such that $e \in \overset{\circ}{A}_1$, $A_n \subset \overset{\circ}{A}_{n+1}$ for $n \geq 1$ and $N = \cup_n A_n$. Let $h_n(a)$ be the Dirichlet solution in A_n with boundary values 1 at e and 0 on ∂A_n, then extended by 0 outside A_n. Then $\{h_n\}$ is an increasing sequence of bounded functions on N, $0 \leq h_n(a) \leq 1$. Let $h(a) = \lim\limits_{n \to \infty} h_n(a)$. The function $h \equiv 1$ if and only if N is a parabolic network. Otherwise it is hyperbolic; notice that in this case $h(a) = R_1^e(a)$.

§ 3. Random walks

A random walk $\{N, P = [p(a,b)]\}$ behaves in some case (when the matrix P is irreducible) similar to an infinite network $\{X, p(a,b)\}$ with the restriction $\sum\limits_{b \sim a} p(a,b)u(b)$. A real-valued function $u(a)$ is said to be upper Δ-function if $Pu(a) \leq u(a)$ for all a. If $u(a)$ is a function such that $u(a) > -\infty$ for all $a \in N$, $Pu(a) \leq u(a)$ is finite at one vertex c, then $u(a)$ is real-valued on N and consequently upper Δ-function. For $u(c) \geq Pu(c)$ implies that $u(a)$ is real-valued for all $a \sim c$; this leads to the conclusion that $u(a)$ is real-valued on N since N is connected.

We write $-\Delta = (I - P)$. The infinite network $\{N, p(a,b)\}$ associated with the random walk $\{N, P\}$ is referred to as a parabolic network if every positive upper Δ-function in $\{N, p(a,b)\}$ is constant; if there exists a non-Δ-function positive upper Δ-function on $\{N, p(a,b)\}$, then it is referred to as a hyperbolic network.

Let us start with a time-homogeneous Markov chain that is a discrete-time stochastic process $\{N_n\}$, $n = 0, 1, 2, \ldots$, where N_n takes values in the state space N with a countable infinite states [17]. For any two states a, b the transition probability from a to b is denoted by $p(a,b) = \operatorname{Prob}\{N_1 = b, N_0 = a\}$. Thus, the set N with the transition numbers $p(a,b)$ can be considered as an infinite network in which a and b are neighbours if and only if $p(a,b) > 0$; at this stage N may or may not be a connected graph. Denote by P the infinite matrix of the transition probabilities $\{p(a,b)\}$. In $\{N, p(a,b)\}$, just as $p(a,b)$ represents the probability that the walker starting at the state b reaches the state a, $p^n(a,b)$ represents the probability that the walker starting at b reaches a in n steps. Actually $p^n(a,b)$ is the entry in the a^{th} column and the b^{th} row of the matrix p^n. Take $p^0 = I$. Let us assume that given any two states a and b, there exist integers m and n such that $p^m(a,b) > 0$ and $p^n(b,a) > 0$; in this case the matrix P is referred to as irreducible. When P is irreducible, the infinite graph $\{N, p(a,b)\}$ is actually connected so that it is an infinite network in the earlier sense. When the matrix P is irreducible, we also refer to $\{N, P\} = \{N, P = [p(a,b)]\}$ as a random walk with the state space N and the transition probability matrix P determined by the process $\{N_n\}$.

Definition 1. *An irreducible Markov chain $\{N_n\}$ on N is said be recurrent* if for each state a, the chain returns to a infinitely often. That is, $\mathrm{Prob}\,\{N_n = a \text{ for infinitely many } n\} = 1$.

Since the transition probabilities matrix is assumed to be irreducible, then starting from a state b the walker can reach any other state a in finite steps. Consequently certain variations in the above definition can be proposed:

(i) suppose e is a fixed state and a is any other state; then $\{N_n\}$ is recurrent if and only if the walker starting from a reaches e infinitely often;

(ii) if the irreducible chain visits a state infinitely often, then it also visits every other state in N infinitely often.

Definition 2. *An irreducible Markov chain $\{N_n\}$ on N is called transient* if it is not recurrent. Thus, transient means that the chain visits any state only a finite number of times and then wanders off to the state at infinity.

Thus, the division of random walks into two groups, recurrent and transient, depends on the situation whether the Markov chain $\{N_n\}$ returns to any starting state infinitely often or only a finite number of times. This distinction is manifested in different forms in the classification of random walks as shown below and we also interpret these results in the context of infinite networks associated with the respective random walks.

The following passage up to the proof of Proposition 4 is mainly based on Lawler [17, Section 2.2]. Fix a state e and assume that $N_0 = e$. Consider the random variable R which gives the total number of visits to e including the initial visit. Write $R = \sum\limits_{n=0}^{m} \chi\{N_n = e\}$ where χ is the characteristic function. When the chain is recurrent, R is identically ∞. That is, if $R_m = \sum\limits_{n=0}^{m} \chi\{N_n = e\}$, then $R_m \to \infty$ when $m \to \infty$. Now the expectation is $E(R_m) = $ $= \sum\limits_{n=0}^{\infty} \mathrm{Prob}\,\{N_n = e\} = \sum\limits_{n=0}^{m} p^n(e,e)$. Hence in the case of recurrence $\sum\limits_{n=0}^{\infty} p^n(e,e) = \infty$.

Note that $R < \infty$ with probability 1 if the chain is transient. In this case the expectation of R is $E(R) = E\left[\sum\limits_{n=0}^{\infty} \chi\{N_n = e\}\right] = \sum\limits_{n=0}^{\infty} \mathrm{Prob}\,\{N_n = e\} = \sum\limits_{n=0}^{\infty} p^n(e,e)$.

Proposition 1. *The Markov chain is transient if and only if $\sum\limits_{n=0}^{\infty} p^n(e,e) < \infty$.*

P r o o f. From the above narrative, if $\sum\limits_{n=0}^{\infty} p^n(e,e) < \infty$ then the chain cannot be recurrent. Conversely, assume that the chain is transient. That is, the chain $\{N_n\}$ returns to e only a finite number of times. Let q be the probability of the first return of $\{N_n\}$ to e. Note that $q \neq 1$ since the chain is transient: if $q = 1$ then with probability 1 the chain always returns to e and by continuing we see that the probability is 1 for the chain to returns to e infinitely often; that is the chain is recurrent.

In the case of transience, $R = 1$ if and only if the chain never returns to e, hence the probability is $1 - q$; and $R = m$ if and only if the chain returns $(m-1)$ times and does not return for the m^{th} time, hence the probability is $q^{m-1}(1-q)$. Consequently, $E(R) = \sum\limits_{m=1}^{\infty} m \cdot \text{Prob}\{R = m\} = \\ = \sum\limits_{m=1}^{\infty} m\,[q^{m-1}(1-q)] = \frac{1}{1-q} < \infty$. Comparing this with the earlier expression for $E(R)$ in the case of transience, we conclude that the Markov chain is transient if and only if

$$\sum_{n=0}^{\infty} p^n(e,e) < \infty. \qquad \square$$

Remark 1. For any state c, the walker can reach e from c in a finite number of steps. Thus, the nature of transience does not depend on the choice of the initially fixed state e. Consequently, the Markov chain $\{N_n\}$ is transient if and only if $\sum\limits_{n=0}^{\infty} p^n(c,c) < \infty$ for any state c. Instead of the circuit probabilities like $p^n(c,c)$, we shall now take up the consideration of $p^n(a,b)$ which is the probability that the walker starting at the state N reaches the state b in n steps. For this, it is easier to consider $\{N, P = [p(a,b)]\}$ either as a random walk or an infinite network as the occasion demands.

Writing $p^n(a,b)$ as $p_b^n(a)$, remark that

$$P\,[p_b^n(a)] = \sum_c p(a,c)p_b^n(c) = \sum_c p(a,c)p^n(c,b) = p^{n+1}(a,b) = p_b^{n+1}(a),$$

since $p_b^n(a)$ denotes the probability that the walker starting from the state a reaches the state b in n steps, then the expression $G_b(a) = \sum\limits_{n=0}^{\infty} p_b^n(a)$ represents the expected number of visits to the state b starting from the state a.

Proposition 2. *If the random walk $\{N, P = [p(a,b)]\}$ is transient, then the infinite network $\{N, p(a,b)\}$ associated with it is hyperbolic.*

P r o o f. We shall actually show that $G_b(a)$ is the Green basis function on the network $\{N, p(a,b)\}$ with Δ-function support at $\{b\}$.

Choose a vertex b in the network N. If $G_b(a) = \sum\limits_{n=0}^{\infty} p_b^n(a)$, then $P\,[G_b(a)] = \sum\limits_{n=1}^{\infty} p_b^n(a) \leq \\ \leq G_b(a)$ so that $G_b(a)$ is a positive upper Δ-function in the network $\{N, p(a,b)\}$.

The function $G_b(a)$ is actually a basis function. For that note that when $G_b(a)$ is real-valued, we can write $G_b(a) - P\,[G_b(a)] = \delta_b(a)$ which is the column vector with entry 1 when $b = a$ and 0 in other entries. Consequently, $-\Delta\,[G_b(a)] = \delta_b(a)$.

If $h \geq 0$ is a Δ-function such that $h(a) \leq G_b(a)$, then we have $h(a) = P^m h(a) \leq \\ \leq P^m\,[G_b(a)] = \sum\limits_{n=m}^{\infty} p_b^n(a)$ which tends to 0 when $m \to \infty$; this shows that $h \equiv 0$. Hence $G_b(a)$ is a basis function which in this case is the Green basis function having $\{b\}$ as its Δ-function support. $\square$

Remark 2. The above theorem can be reformulated: A random walk $\{N, P = [p(a,b)]\}$ is recurrent if the associated infinite network $\{N, p(a,b)\}$ is parabolic. Conversely, if the Markov chain is reversible, then the parabolicity of the network implies that the random walk is recurrent. ("Reversible" means that there exists a real-valued function $\phi(a) > 0$ such that $\phi(a)p(a,b) = \varphi(b)p(b,a)$ for any two states a, b.) This converse can be deduced [2, Theorem 3.3] from McGuinness [1, p. 90]. See the very important papers of Nash–Williams [6] and Lyons [7] in this context.

§ 4. Infinite trees

A connected graph is called a tree if there is no cycle in it, that is there is no closed path $\{a_1, a_2, \ldots, a_n, a_1\}$ with $n \geq 3$. Thus, in an infinite tree T, if a, b are any two vertices then there exists a unique path connecting a to b. Suppose a random walk $\{T, P = [p(a,b)]\}$ is defined in the infinite tree T.

Fix a vertex e in T. Then for any a in T, if $\{e, a_1, a_2, \ldots, a_n = a\}$ is the unique path connecting e to a, write $\phi(a) = \frac{p(e,a_1)p(a_1,a_2)\ldots p(a_{n-1},a)}{p(a,a_{n-1})p(a_{n-1},a_{n-2})\ldots p(a_1,e)}$. Note that if $b \sim a$, then $\phi(a)p(a,b) = = \phi(b)p(b,a)$. Hence $\{T, P\}$ is reversible, which leads to the conclusion: The random walk $\{T, P = [p(a,b)]\}$ on the infinite tree T is recurrent if and only if the associated network $\{T, p(a,b)\}$ is parabolic.

Proposition 3. *Let $\{N, P = [p(a,b)]\}$ be a random walk. Suppose there exists a function v defined outside a finite set A in N such that $(I-P)v(a) \geq 0$ at every $a \in N/A$ and $\lim\limits_{a\to\infty} v(a) = = \infty$. Then the random walk is recurrent.*

P r o o f. With the existence of such a function $v(a)$, the network $\{N, p(a,b)\}$ has to be parabolic. Otherwise, for each vertex $b \in N$ there exists the Green basis function $G_b(a)$ which is bounded and $(I-P)G_b(a) = \delta_b(a)$. Choose a large finite set E, $E_0 \supset A$. Let h be the Dirichlet solution on E with boundary values v on ∂E. Let v_1 be the function on N such that $v_1 = h$ on E and $v_1 = v$ on (N/E). Define for $a \in N$, $v_2(a) = v_1(a) + \sum\limits_{b\in\partial E} \Delta V_1(b) G_b(a)$.

Then for $a \in \partial\overset{\circ}{E}$, $(I-P)v_2(a) = 0$; for $a \in (N/E)$, $(I-P)v_1(a) \geq 0$; for $a \in \partial E$, $(I-P)v_2(a) = (I-P)v_1(a) + (-)(I-P)v_1(a) = 0$. Thus, $(I-P)v_2(a) \geq 0$ on N. Now $G_b(a)$ is bounded on N, so that $\lim\limits_{a\to\infty} v_2(a) = \infty$. But this is not possible by the Minimum Principle for v_2. Consequently, the assumption that $\{N, p(a,b)\}$ is not parabolic is invalid. So the random walk $\{N, P\}$ is recurrent.

Let us consider now a random walk $\{T, P = [p(a,b)]\}$ on an infinite tree T. Fixing a vertex $e \in T$, let us measure distance from e. Remark that T is reversible and that for any $a \in T$, $|a| = n$, there is one neighbour $\tilde{a}$, $|\tilde{a}| = n-1$; other neighbours b_i are at a distance $|b_i| = n+1$. □

Proposition 4. *Let $\{T, P = [p(a,b)]\}$ be a random walk on an infinite tree. Measure distances in T from a fixed vertex e. If $p(a,\tilde{a}) \geq \frac{1}{2}$ for all a, then $\{T, P\}$ is recurrent. If $p(a,\tilde{a}) < \frac{1}{2}$, then $\{T, P\}$ is transient.*

P r o o f. Consider the function $f(n) = \left(\frac{\alpha}{1-\alpha}\right)^n$, $0 < \alpha < 1$, at any a, $|a| = n \geq 1$, we have $(I-P)f(n) = -\left[1 - 2p(a,\tilde{a})\right] \frac{2\alpha-1}{1-\alpha} \left[\frac{\alpha}{1-\alpha}\right]^n$.

1. Suppose $p(a,\tilde{a}) > \frac{1}{2}$ for all a. Then take $1 > \alpha > \frac{1}{2}$. In this case $\frac{\alpha}{1-\alpha} > 1$. Hence $(I-P)f(n) > 0$ outside e, and $f(n) \to \infty$ at the point at infinity. Hence by Proposition 3, $\{T, P\}$ is recurrent.

2. Suppose $p(a,\tilde{a}) < \frac{1}{2}$ for all a. Then take $0 < \alpha < \frac{1}{2}$. Hence $(I-P)f(n) > 0$. At e $(I-P)f(e) = -\left[\frac{\alpha}{1-\alpha} - 1\right] > 0$. Hence $f(n)$ is a positive upper Δ-function tending to 0 at infinity, hence a basis function, so that $\{T, P\}$ is transient. (Remark 2 following Proposition 2.)

3. The case $p(a,\tilde{a}) = \frac{1}{2}$: for the function $s(a) = n$ when $|a| = n \geq 1$, $(I-P)s(a) > 0$; moreover, $s(a) \to \infty$. Hence $\{T, P\}$ is recurrent.

Let $\{N, p(a,b)\}$ be the infinite network associated with the random walk $\{N, P = [p(a,b)]\}$. Let us recall the notion of reduced functions in the network [3]. If $s(a)$ is a non-negative upper Δ-function defined on a set E and A is a subset in the interior E^0 of E, then $\left[R_s^A(a)\right]_E = \inf_{u \in \Im} u(a)$ where $\Im$ is the family of non-negative upper Δ-functions $u(a)$ on E such that $u(a) \geq s(a)$ on A. $\square$

Example 1. Let T be a homogeneous tree of degree 2 and the transition probability $p(a, \tilde{a}) = \frac{q+1}{2}$. Consider a function $s(a) = 2^{-1+|a|}$. Note that for any vertex a in T, $|a| = n \geq 1$. Here $|a|$ represents the distance between the root vertex.

P r o o f.

$$\begin{aligned}
\Delta s(a) &= \frac{q+1}{2}[2^{-1+(n-1)} - 2^{-1+n}] + \frac{q-1}{2}[2^{-1+(n+1)} - 2^{-1+n}] \\
&= \frac{q+1}{2}[2^{-1+n-1} - 2^{-1+n}] + \frac{q-1}{2}[2^{-1+n+1} - 2^{-1+n}] \\
&= \frac{q+1}{2}[2^{n-2} - 2^{-1+n}] + \frac{q-1}{2}[2^{n} - 2^{-1+n}] \leq 0.
\end{aligned}$$

If $p(a, \tilde{a}) \geq \frac{1}{2}$ for all a, then $\{T, P\}$ is recurrent. If $p(a, \tilde{a}) < \frac{1}{2}$, then $\{T, P\}$ is transient. $\square$

Lemma 1. *Let E be a finite set $e \in \overset{\circ}{E}$. Then $[R_1^e(a)]_E$ is the Dirichlet solution in E with boundary values 1 at e and 0 at each vertex in ∂E.*

P r o o f. Let $\varphi(a)$ be the unique Dirichlet solution on E with boundary values 1 at e and 0 on ∂E. Then $\varphi(a) \geq [R_1^e]_E$ on E. Since $R_1^e(a)$ is a non-negative upper Δ-function on E with values 1 at e and 0 on ∂E, by the construction of the Dirichlet solution we have $\varphi(a) \geq [R_1^e(a)]_E$. This proves $\varphi(a) = [R_1^e(a)]_E$ on E. $\square$

Lemma 2. *Let $e \in \overset{\circ}{E}$ where E is a finite set. Then the probability that the walker starting at a state $a \in \overset{\circ}{E}$ goes outside $\overset{\circ}{E}$ before ever coming back to e is $1 - [R_1^e(a)]_E$.*

P r o o f. Let $\varphi(a)$ be the probability that the walker starting at a reaches e before visiting any state in ∂E. Then $\varphi(e) = 1$, $\varphi(c) = 0$ for $c \in \partial E$; moreover, for any $a \in \overset{\circ}{E}$, we have $\varphi(a) = \sum_{b \sim a} p(a,b)\varphi(b)$. This means $(I - P)\varphi(a) = 0$. That is, $\varphi(a)$ is harmonic on E with boundary values $\varphi(e) = 1$, $\varphi(c) = 0$ on ∂E. Hence by the above lemma 1, $\varphi(a) = [R_1^e(a)]_E$. This shows that the walker starting at $a \in \overset{\circ}{E}$ goes outside $\overset{\circ}{E}$ before ever coming to e with the probability $1 - \varphi(a) = 1 - [R_1^e(a)]_E$. $\square$

Theorem 1. *In the random walk $\{N, P = [p(a,b)]\}$, the probability that the walker starting at the state a goes off to infinity A without visiting e is $1 - \varphi(a) = 1 - [R_1^e(a)]$ which is defined with reference to the associated network $\{N, p(a,b)\}$.*

P r o o f. Let $\{E_n\}$ be an increasing sequence of finite sets such that $N = \cup E_n$. For any a in N, if $a \in E_m$ then $[R_1^e(a)]_n$ (which represents the reduced function with respect to the finite set E_n) is defined for $n \geq m$ and is an increasing sequence of upper Δ-functions. Since $[R_1^e(a)]_n \leq R_1^e(a)$, then $v(a) = \sup_n [R_1^e(a)]_n$ is an upper Δ-function on N and $v(a) \leq R_1^e(a)$. On the other hand, since $v(a)$ is an upper Δ-function on N and $v(e) = 1$, we have $v(a) \geq R_1^e$ also. Thus, $R_1^e(a) = v(a) = \lim_{n \to \infty} [R_1^e(a)]_n$.

Now the probability that the walker starting at the state x and going off to infinity A without visiting e is the limiting value of $1 - [R_1^e(a)]_n$ when $n \to \infty$ which is $1 - R_1^e(a)$. $\square$

Corollary 1. *In the random walk $\{N, P = [p(a,b)]\}$, the probability that the walker starting at e, after leaving e never returns to e is $(I-P)R_1^e$.*

P r o o f. The probability (from the above Theorem 1) is

$$\sum_{b\sim e} p(e,b)\left[1 - R_1^e(b)\right] = \sum_{b\sim e} p(e,b)\left[R_1^e(e) - R_1^e(b)\right] = -\Delta\left[R_1^e(e)\right] = (I-P)R_1^e(e). \qquad \square$$

Some remarks on the reduced function $R_1^e(a)$ in the infinite network $\{N, p(a,b)\}$.

1. The network N is parabolic if and only if $R_1^e \equiv 1$ on N.

2. The network N is hyperbolic if and only if $R_1^e(a)$ is a basis function on N.

3. As in [3, Section 3.2], for each $e_i \sim e$, denote by $[e, e_i]$ the subset $[e, e_i] = \{a\colon$ there exists a path joining a to e that passes through $e_i\}$; e_i and e are assumed to be in $[e, e_i]$. Note that if e_i, e_j ate two neighbours of e, then either $[e, e_i]$ and $[e, e_j]$ are two subsets having e as the only common vertex or $[e, e_i] = [e, e_j]$. The subset $[e, e_i]$ is called an S-domain if 0 is the only bounded function $h(a)$ on $[e, e_i]$ such that $h(e) = 0$ and $-\Delta h(c) = (I-P)h(c) = 0$ for any $c \neq e$. A subset $[e, e_j]$ is called a P-domain if it is not an S-domain. If a set $[e, e_i]$ contains only a finite number of vertices, then it is necessarily an S-domain. The network $\{N, p(a,b)\}$ is parabolic if and only if all the subsets $[e, e_i]$ are S-domains. It is hyperbolic if and only if at least one $[e, e_i]$ is a P-domain; in this case there may be other subsets that are S-domains.

4. If the random walk $\{N, P = [p(a,b)]\}$ is transient, it has been seen that $G_e(a) = \sum_{n=0}^{\infty} p^n(a,e)$ represents the expected number of visits to the state e starting from the state a. The function $G_e(a)$ can also be interpreted as the Green function in the hyperbolic network $\{N, p(a,b)\}$ with Δ-function support at e. Now (see [3, Corollary 3.3.7]), $G_e(a) \leq G_e(e)$ for all $a \in N$; in fact, $G_e(a) = G_e(e)R_1^e(a)$.

5. It can also be mentioned that in the case of a transient random walk, if A is a finite set in N, then R_1^A denotes the probability that the walker starting at the state a to reaches a state in A before wandering off to infinity.

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Δ-функции на рекуррентных случайных блужданиях

Ключевые слова: параболические сети, решения Дирихле, выметание, рекуррентные случайные блуждания.

УДК 519.21, 531.26

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Если случайное блуждание на бесконечном счетном пространстве состояний обратимо, то известны необходимые и достаточные условия для того, чтобы это блуждание было рекуррентным. Если отбросить условие обратимости, то, используя дискретные решения Дирихле и выметание (понятия, известные из теории потенциала), можно частично установить некоторые из приведенных выше результатов, касающихся повторяемости и переходности случайного блуждания.

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