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UPPER AND LOWER DIRECTIONAL DERIVATIVE SETS AND DIFFERENTIALS OF THE SET VALUED MAPS

In this paper directional derivative sets and differentials of a given set valued map are studied. Different type relations between directional derivative sets and differentials of a set valued map are specified. It is established that every compact subset of lower derivative set can be used for lower approximation of given set valued map. Upper and lower contingent cones of some plane sets are calculated and compared.

Keywords: set valued map, contingent cone, differential, directional derivative set, Hausdorff deviation.

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Introduction

Set valued maps appear in the mathematical models of many problems arising in physics, mechanics, economics, biology, etc. They are very adequate tools for description and investigation of the problems springing up in the theory and applications, especially in the optimization theory, optimal control theory, game theory problems (see, e. g., [1–6] and references therein). In studying of these problems one often has to deal with differential or directional derivative sets of the given set valued maps which essentially simplify the description of the considered processes and expression of the obtained results (see, e. g., [7–11]). Unfortunately there is no unified approach for definition of the differential and derivative for the set valued map. Depending on the character of the considered problem, the various appropriate types of differential and derivative concepts for the set valued maps are applied (see, e. g., [7–17]). In general, the differential notions of the set valued maps are based on the various types of tangent and contingent cones. In this paper, the upper and lower differentials of the set valued map introduced in [12] and [18] are used. The aforementioned concepts are based on the upper and lower contingent cones and are applied for investigation of many problems of the set valued and nonsmooth analysis (see, e. g., [1, 5, 10, 12, 13, 16, 18, 19]).

The definition of directional derivative set is different from the definition of differential, but they are also closely connected with the concept of contingent cones. In the presented paper, the relations between differentials (upper and lower) and directional derivative sets (upper and lower) of a set valued map are studied.

The paper is organized as follows: In Section 1, the definitions of upper and lower contingent cones of the sets is formulated. The upper and lower contingent cones of some given sets on the plane is calculated and compared (Example 1 and Example 2). In Section 2 the directional upper and lower derivative sets of a set valued map are defined and the relations between directional upper (lower) derivative sets and upper (lower) differentials are studied. It is shown that if the set valued map is not locally Lipschitz continuous, then lower derivative set in the direction p and the value of the lower differential at p does not coincide (Example 3). For scalar variable set valued maps, it is proved that upper derivative set in the direction p and the value of the upper differential at p are equal (Theorem 1). In Section 3 the properties of the compact subsets of the directional derivative sets and differentials are investigated. The Hausdorff deviation of the cone generated by a compact subset of the lower directional derivative set from the given set valued map is estimated (Theorem 2, Corollary 1).

§ 1. Upper and lower contingent cones

Let us give the definitions of upper and lover contingent cones.

Definition 1 (see [18, 19]). Let X be a Banach space, $K \subset X$ and $x \in X$. The sets

$$T_K^U(x) = \left\{ u \in X \colon \liminf_{\delta \to 0^+} \frac{1}{\delta} d(x + \delta u, K) = 0 \right\}$$

and

$$T_K^L(x) = \left\{ u \in X \colon \lim_{\delta \to 0^+} \frac{1}{\delta} d(x + \delta u, K) = 0 \right\}$$

are called the upper and lower contingent cone of the set K at $x \in X$ respectively, where $d(y, K) = \inf_{z \in K} \|y - z\|$, i. e., $d(y, K)$ is the distance from the point y to the set K.

$T_K^U(x)$ and $T_K^L(x)$ are closed cones in the space X and $T_K^L(x) \subset T_K^U(x)$. It is obvious that $u \in T_K^U(x)$ if and only if there exist sequences $\{\delta_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ ($s_i \in X$) such that $\delta_i \to 0^+$ and $s_i \to 0$ as $i \to +\infty$ and the inclusion $x_i = x + \delta_i u + \delta_i s_i \in K$ is satisfied for every $i = 1, 2, \ldots$.

Similarly, $u \in T_K^L(x)$ if and only if there exist $\delta_* > 0$ and $s(\cdot) \colon (0, \delta_*] \to X$ such that $s(\delta) \to 0$ as $\delta \to 0^+$ and the inclusion $x(\delta) = x + \delta u + \delta s(\delta) \in K$ is verified for every $\delta \in (0, \delta_*]$.

Now, let us compare the upper and lower contingent cones of some sets given on the plane.

Example 1. Let the set $K \subset \mathbb{R}^2$ be given as

$$K = \left\{ \left(\frac{1}{n}, \frac{1}{n} \right) \in \mathbb{R}^2 \colon n \in \mathbb{N} \right\} \bigcup \{(0, 0)\}, \tag{1.1}$$

where $\mathbb{N} = \{1, 2, \ldots\}$.

Let us show that

$$T_K^L(0, 0) = T_K^U(0, 0) = \{(\alpha, \alpha) \in \mathbb{R}^2 \colon \alpha \geq 0\}. \tag{1.2}$$

First of all, we prove that $(1, 1) \in T_K^L(0, 0)$. Choose an arbitrary sequence $\{\delta_i\}_{i=1}^{\infty}$ such that $\delta_i \to 0^+$ as $i \to +\infty$. Then for each δ_i there exists $m_i \in \mathbb{N}$ such that

$$\delta_i \in \left(\frac{1}{m_i + 1}, \frac{1}{m_i} \right]. \tag{1.3}$$

Since $\delta_i \to 0^+$ as $i \to +\infty$, then $m_i \to +\infty$ as $i \to +\infty$. Thus, (1.3) implies

$$\begin{aligned}
\lim_{i \to \infty} \frac{1}{\delta_i} d\left((0,0) + \delta_i(1,1), K\right) &= \lim_{i \to \infty} \frac{1}{\delta_i} d\left((\delta_i, \delta_i), K\right) \\
&\leq \lim_{i \to \infty} \frac{1}{\delta_i} d\left(\left(\frac{1}{m_i}, \frac{1}{m_i} \right), \left(\frac{1}{m_i + 1}, \frac{1}{m_i + 1} \right) \right) = \lim_{i \to \infty} \frac{1}{\delta_i} \frac{\sqrt{2}}{m_i(m_i + 1)} \\
&\leq \lim_{i \to \infty} (m_i + 1) \frac{\sqrt{2}}{m_i(m_i + 1)} = \lim_{i \to \infty} \frac{\sqrt{2}}{m_i} = 0.
\end{aligned}$$

Since the sequence $\{\delta_i\}_{i=1}^{\infty}$ is arbitrarily chosen, we conclude that

$$\lim_{\delta \to 0^+} \frac{1}{\delta} d\left((0,0) + \delta(1,1), K\right) = 0$$

which yields that $(1,1) \in T_K^L(0,0)$. Since $(1,1) \in T_K^L(0,0)$ and $T_K^L(0,0) \subset \mathbb{R}^2$ is a cone, then we obtain that $(\alpha,\alpha) \in T_K^L(0,0)$ for every $\alpha \geq 0$. Thus the inclusion

$$\bigl\{(\alpha,\alpha) \in \mathbb{R}^2 \colon \alpha \geq 0\bigr\} \subset T_K^L(0,0) \tag{1.4}$$

is verified.

Now, we choose an arbitrary $(\alpha,\beta) \in T_K^U(0,0)$. By virtue of Definition 1 there exist sequences $\{\delta_i\}_{i=1}^{\infty}$ and $\{(p_i,q_i)\}_{i=1}^{\infty}$ such that $\delta_i \to 0^+$, $(p_i,q_i) \to (0,0)$ as $i \to +\infty$ and

$$(x_i,y_i) = (0,0) + \delta_i(\alpha,\beta) + \delta_i(p_i,q_i) \in K \tag{1.5}$$

for every $i = 1,2,\ldots$. According to (1.1) we have that $x_i = y_i$ for every $i = 1,2,\ldots$. It follows from (1.5) that

$$\alpha + p_i = \beta + q_i \tag{1.6}$$

for every $i = 1,2,\ldots$. Since $p_i \to 0$, $q_i \to 0$ as $i \to +\infty$, then we obtain from (1.6) that $\alpha = \beta$, and hence again by virtue of (1.6) $p_i = q_i$ for every $i = 1,2,\ldots$. Concluding, we obtain from (1.5) that

$$(\alpha,\alpha) + (p_i,p_i) \in \frac{1}{\delta_i}K \tag{1.7}$$

for every $i = 1,2,\ldots$. Since $p_i \to 0$ as $i \to +\infty$, then (1.1) and (1.7) yield that $\alpha \geq 0$. Thus, for arbitrarily chosen $(\alpha,\beta) \in T_K^U(0,0)$ we have that $\alpha = \beta$ and $\alpha \geq 0$. This implies that

$$T_K^U(0,0) \subset \bigl\{(\alpha,\alpha) \in \mathbb{R}^2 \colon \alpha \geq 0\bigr\}. \tag{1.8}$$

Since $T_K^L(0,0) \subset T_K^U(0,0)$, then (1.4) and (1.8) yields the validity of equality (1.2).

Now we present an example which illustrates that lover and upper contingent cones not always coincide.

Example 2. Let the set $\Omega \subset \mathbb{R}^2$ be defined as

$$\Omega = \left\{\left(\frac{1}{(2n)!}, \frac{1}{(2n)!}\right) \colon n \in \mathbb{N}\right\} \bigcup \left\{(0,0)\right\}. \tag{1.9}$$

Let us show that $T_\Omega^U(0,0) \not\subset T_\Omega^L(0,0)$.

At first, it will be proved that $(1,1) \notin T_\Omega^L(0,0)$. By virtue of (1.9) we have

$$\begin{aligned} d\left(\left(\frac{1}{(2k+1)!}, \frac{1}{(2k+1)!}\right), \Omega\right) &= \min\left\{\left\|\left(\frac{1}{(2k+1)!}, \frac{1}{(2k+1)!}\right) - \left(\frac{1}{(2k)!}, \frac{1}{(2k)!}\right)\right\|,\right. \\ &\qquad \left.\left\|\left(\frac{1}{(2k+1)!}, \frac{1}{(2k+1)!}\right) - \left(\frac{1}{(2k+2)!}, \frac{1}{(2k+2)!}\right)\right\|\right\} \\ &= \min\left\{\sqrt{2}\cdot\frac{2k}{(2k+1)!}, \sqrt{2}\cdot\frac{2k+1}{(2k+2)!}\right\} = \sqrt{2}\cdot\frac{2k+1}{(2k+2)!} \end{aligned}$$

for every $k = 1,2,\ldots$. Now let us choose a sequence $\{\delta_k\}_{k=1}^{\infty}$, where $\delta_k = \dfrac{1}{(2k+1)!}$. The last equality implies that

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{\delta_k} d\bigl((0,0) + \delta_k(1,1), \Omega\bigr) &= \lim_{k\to\infty} (2k+1)! \cdot d\left(\left(\frac{1}{(2k+1)!}, \frac{1}{(2k+1)!}\right), \Omega\right) \\ &= \lim_{k\to\infty} \sqrt{2}\cdot(2k+1)!\cdot\frac{2k+1}{(2k+2)!} = \sqrt{2}\cdot\lim_{k\to\infty}\frac{2k+1}{2k+2} = \sqrt{2} > 0, \end{aligned}$$

and hence $(1,1) \notin T^L_\Omega(0,0)$.

Now let us show that $(1,1) \in T^U_\Omega(0,0)$. Assume that $\delta_k = \dfrac{1}{(2k)!}$, $k = 1, 2, \ldots$. Then

$$\liminf_{\delta \to 0^+} \frac{1}{\delta} d\big((0,0) + \delta(1,1), \Omega\big) \le \lim_{k\to\infty} \frac{1}{\delta_k} d\big((\delta_k, \delta_k), \Omega\big)$$
$$= \lim_{k\to\infty} (2k)! \cdot d\left(\left(\frac{1}{(2k)!}, \frac{1}{(2k)!}\right), \Omega\right) = 0,$$

and hence $(1,1) \in T^U_\Omega(0,0)$. Since $(1,1) \notin T^L_\Omega(0,0)$, we have that $T^U_\Omega(0,0) \not\subset T^L_\Omega(0,0)$.

Note that similarly to Example 1 it is possible to show that

$$T^L_\Omega(0,0) = \{(0,0)\}, \quad T^U_\Omega(0,0) = \big\{(\alpha,\alpha) \in \mathbb{R}^2 \colon \alpha \ge 0\big\}.$$

§ 2. Upper and lower directional derivative sets and differentials

In this section upper and lower differentials and directional derivative sets of the set valued maps are investigated. The graph of the set valued map $F(\cdot)\colon X \rightsquigarrow Y$ is denoted by $grF(\cdot)$ and is defined as $grF(\cdot) = \{(x,y) \in X \times Y \colon y \in F(x)\}$. From now on it will be assumed that X and Y are Banach spaces.

Let us formulate the definitions of upper and lower differentials of a given set valued map $F(\cdot)$.

Definition 2 (see [12]). Let $F(\cdot)\colon X \rightsquigarrow Y$ be a set valued map, $(x,y) \in X \times Y$. The set valued map $D^U F(x,y)|(\cdot)\colon X \rightsquigarrow Y$ satisfying the equality

$$grD^U F(x,y)|(\cdot) = T^U_{grF(\cdot)}(x,y)$$

is called the upper differential of the set valued map $F(\cdot)$ at the point (x,y) where $T^U_{grF(\cdot)}(x,y)$ is upper contingent cone of the set $grF(\cdot)$ at the point (x,y).

Definition 3 (see [18]). Let $F(\cdot)\colon X \rightsquigarrow Y$ be a set valued map, $(x,y) \in X \times Y$. The set valued map $D^L F(x,y)|(\cdot)\colon X \rightsquigarrow Y$ satisfying the equality

$$grD^L F(x,y)|(\cdot) = T^L_{grF(\cdot)}(x,y)$$

is called the lower differential of the set valued map $F(\cdot)$ at the point (x,y) where $T^L_{grF(\cdot)}(x,y)$ is lower contingent cone of the set $grF(\cdot)$ at the point (x,y).

Now, let us formulate definitions of the upper and lower directional derivative sets of a given set valued map.

Definition 4. Let $F(\cdot)\colon X \rightsquigarrow Y$ be a set valued map, $(x,y) \in X \times Y$ and $p \in X \setminus \{0\}$. The set $\dfrac{\partial^U F(x,y)}{\partial p}$ defined by

$$\frac{\partial^U F(x,y)}{\partial p} = \left\{u \in Y \colon \liminf_{\delta\to 0^+} \frac{1}{\delta} d\big(y + \delta u, F(x+\delta p)\big) = 0\right\}$$

is called upper derivative set of the set valued map $F(\cdot)$ at the point (x,y) in the direction p.

Definition 5. Let $F(\cdot)\colon X \rightsquigarrow Y$ be a set valued map, $(x, y) \in X \times Y$ and $p \in X \setminus \{0\}$. The set $\dfrac{\partial^L F(x,y)}{\partial p}$ defined by

$$\frac{\partial^L F(x,y)}{\partial p} = \left\{ u \in Y\colon \lim_{\delta \to 0^+} \frac{1}{\delta} d\big(y + \delta u, F(x + \delta p)\big) = 0 \right\}$$

is called lower derivative set of the set valued map $F(\cdot)$ at the point (x, y) in the direction p.

It is obvious that for given set valued $F(\cdot)\colon X \rightsquigarrow Y$ the inclusions

$$D^L F(x,y)|(p) \subset D^U F(x,y)|(p)\,, \quad \frac{\partial^L F(x,y)}{\partial p} \subset \frac{\partial^U F(x,y)}{\partial p}$$

are satisfied for every $(x, y) \in X \times Y$ and $p \in X \setminus \{0\}$.

Let us give definition of the locally Lipschitz continuity of a set valued map.

Definition 6. The set valued map $F(\cdot)\colon X \rightsquigarrow Y$ is said to be locally Lipschitz continuous if for each $x \in X$ there exist $L_x \geq 0$ and $r_x > 0$ such that for every $y \in B(x, r_x)$ and $z \in B(x, r_x)$ the inequality

$$h\left(F(y), F(z)\right) \leq L_x \cdot d(y, z)$$

is satisfied where $h\left(F(y), F(z)\right)$ denotes the Hausdorff distance between the sets $F(y)$ and $F(z)$, $B(x, r_x) = \{v \in X\colon \|v - x\| < r_x\}$, $d(y, z)$ stands for the distance between the vectors y and z.

The following proposition characterizes lower and upper derivative sets and differentials of the set valued maps.

Proposition 1. *Let $F(\cdot)\colon X \rightsquigarrow Y$ be a set valued map, $(x, y) \in X \times Y$. Then for every $p \in X \setminus \{0\}$ the inclusions*

$$\frac{\partial^L F(x,y)}{\partial p} \subset D^L F(x,y)|(p), \quad \frac{\partial^U F(x,y)}{\partial p} \subset D^U F(x,y)|(p)$$

are verified.

If $F(\cdot)\colon X \rightsquigarrow Y$ is a locally Lipschitz continuous set valued map, then for every $p \in X \setminus \{0\}$ the equalities

$$\frac{\partial^L F(x,y)}{\partial p} = D^L F(x,y)|(p), \quad \frac{\partial^U F(x,y)}{\partial p} = D^U F(x,y)|(p)$$

hold.

Note that if $F(\cdot)$ is not a locally Lipschitz continuous set-valued map, then the equality $\dfrac{\partial^L F(x,y)}{\partial p} = D^L F(x,y)|(p)$ is not valid.

Example 3. Let $X = Y = \mathbb{R}$ and set-valued map $F(\cdot)\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined as

$$F(x) = \begin{cases} x \cdot \sin \dfrac{1}{x} & \text{if } \ x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{if } \ x = 0. \end{cases} \tag{2.1}$$

The map $F(\cdot)\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ defined by (2.1) is not locally Lipschitz continuous on $\mathbb{R}$. Since

$$\frac{1}{\delta} d(0 + \delta \cdot 0, F(0 + \delta \cdot 1) = \frac{1}{\delta} d(0, F(\delta)) = \frac{1}{\delta} \cdot \delta \left| \sin \frac{1}{\delta} \right| = \left| \sin \frac{1}{\delta} \right|$$

for every $\delta > 0$, then we have that $\lim\limits_{\delta\to 0^+} \frac{1}{\delta} d(0+\delta\cdot 0, F(0+\delta\cdot 1)$ does not exist, and hence

$$0 \notin \frac{\partial^L F(0,0)}{\partial 1}. \tag{2.2}$$

Now let us show that $0 \in D^L F(0,0)|(1)$. Choose an arbitrary sequence $\{\delta_k\}_{k=1}^{\infty}$ such that $\delta_k \to 0^+$ as $k \to +\infty$. Then for each k there exists i_k such that $\delta_k \in \left(\frac{1}{\pi(i_k+1)}, \frac{1}{\pi i_k}\right]$. Since $\delta_k \to 0^+$ as $k \to +\infty$, then $i_k \to +\infty$ as $k \to +\infty$. It is obvious that

$$\frac{1}{\delta_k} < \pi(i_k+1) \tag{2.3}$$

and

$$d\big((\delta_k, 0), grF(\cdot)\big) \le \frac{1}{\pi i_k} - \frac{1}{\pi(i_k+1)}. \tag{2.4}$$

From (2.3) and (2.4) it follows

$$\lim_{k\to\infty} \frac{1}{\delta_k} d\big((0,0)+\delta_k(1,0), grF(\cdot)\big) = \lim_{k\to\infty} \frac{1}{\delta_k} d((\delta_k,0), grF(\cdot))$$
$$\le \lim_{k\to\infty} \pi(i_k+1)\left[\frac{1}{\pi i_k} - \frac{1}{\pi(i_k+1)}\right] = \lim_{k\to\infty} \frac{1}{i_k} = 0. \tag{2.5}$$

Since $\{\delta_k\}_{k=1}^{\infty}$ is arbitrarily chosen, then (2.5) implies that

$$\lim_{\delta\to 0^+} \frac{1}{\delta} d\big((0,0)+\delta(1,0), grF(\cdot)\big) = 0,$$

and consequently $(1,0) \in T^L_{grF(\cdot)}(0,0)$. This means that

$$0 \in D^L F(0,0)|(1). \tag{2.6}$$

(2.2) and (2.6) yield that

$$D^L F(0,0)|(1) \ne \frac{\partial^L F(0,0)}{\partial 1}.$$

Theorem 1. *Let $F\colon \mathbb{R} \rightsquigarrow Y$ be a set valued map, $(x,y) \in \mathbb{R}\times Y$. Then for each $p \in \mathbb{R}\setminus\{0\}$ the equality*

$$\frac{\partial^U F(x,y)}{\partial p} = D^U F(x,y)|(p)$$

holds.

P r o o f. By virtue of Proposition 1 we have

$$\frac{\partial^U F(x,y)}{\partial p} \subset D^U F(x,y)|(p). \tag{2.7}$$

Let us prove that

$$D^U F(x,y)|(p) \subset \frac{\partial^U F(x,y)}{\partial p}. \tag{2.8}$$

If $D^U F(x,y)|(p) = \emptyset$, then (2.7) implies that $\frac{\partial^U F(x,y)}{\partial p} = \emptyset$ and the equality $\frac{\partial^U F(x,y)}{\partial p} = D^U F(x,y)|(p)$ is satisfied trivially.

Let $D^U F(x,y)|(p) \neq \emptyset$. Choose an arbitrary $v \in D^U F(x,y)|(p)$. Then $(p,v) \in T^U_{grF(\cdot)}(x,y)$. According to Definition 1 there exist sequences $\{\delta_k\}_{k=1}^{\infty}$ and $\{(s_k,q_k)\}_{k=1}^{\infty}$ such that $\delta_k \to 0^+$, $(s_k,q_k) \to (0,0)$ as $k \to +\infty$ and

$$(x,y) + \delta_k(p,v) + \delta_k(s_k,q_k) \in grF(\cdot) \tag{2.9}$$

for every $k = 1,2,\dots$. Let

$$\beta_k = \frac{p+s_k}{p} \cdot \delta_k \tag{2.10}$$

Since $p \neq 0$ and $s_k \to 0$ as $k \to +\infty$, then without loss of generality it is possible to assume that $\beta_k > 0$ for every $k = 1,2,\dots$. From (2.10) it follows that $\beta_k \to 0^+$ as $k \to +\infty$.

Now, from (2.9) and (2.10) we obtain that

$$\left(x + \beta_k p, y + \frac{p}{p+s_k}\beta_k v + \frac{p}{p+s_k}\beta_k q_k\right) \in grF(\cdot) \tag{2.11}$$

for every $k = 1,2,\dots$. Denote

$$b_k = \frac{p}{p+s_k}v - v + \frac{p}{p+s_k}q_k. \tag{2.12}$$

It is obvious that $b_k \to 0$ as $k \to +\infty$. Relations (2.11) and (2.12) imply that

$$(x + \beta_k p, y + \beta_k v + \beta_k b_k) \in grF(\cdot)$$

and hence

$$y + \beta_k v + \beta_k b_k \in F(x + \beta_k p) \tag{2.13}$$

for every $k = 1,2,\dots$. Inclusion (2.13) yields that

$$\lim_{k\to\infty} \frac{1}{\beta_k} d(y + \beta_k v, F(x+\beta_k p)) \leq \lim_{k\to\infty} \frac{1}{\beta_k}[d(y+\beta_k v, y + \beta_k v + \beta_k b_k)$$
$$+ d(y + \beta_k v + \beta_k b_k, F(x+\beta_k p)] = \lim_{k\to\infty}\frac{1}{\beta_k}\beta_k \|b_k\| = 0,$$

and consequently

$$\liminf_{\beta\to 0^+} \frac{1}{\beta} d(y + \beta v, F(x+\beta p)) = 0.$$

The last equality implies that

$$v \in \frac{\partial^U F(x,y)}{\partial p}.$$

Since $v \in D^U F(x,y)|(p)$ is arbitrarily chosen, we obtain validity of the inclusion (2.8). Inclusions (2.7) and (2.8) complete the proof. □

Remark 1. In Corollary 26.1 of book [16] it is proved that for the set valued map $F(\cdot)\colon X \rightsquigarrow Y$ the equality

$$D^U F(x,y)|(p) = \left\{ u \in Y\colon \liminf_{\delta\to 0^+,\, p'\to p} d\left(u, \frac{1}{\delta}\left(F\left(x+\delta p'\right) - y\right)\right) = 0 \right\} \tag{2.14}$$

holds where X and Y are the Banach spaces.

Note that the Theorem 1 simplifies the aforementioned corollary for the case $X = \mathbb{R}$, stating that if $X = \mathbb{R}$, then in equality (2.14) there is no need to take the limit $p' \to p$ where $p \neq 0$.

§ 3. Properties of the compact subsets of the directional derivative sets

In this section, the relation between a compact subset of the lower directional derivative set and the given set valued map is studied.

The Hausdorff deviation of the set E from the set D is denoted by $h^*(E, D)$ and defined as

$$h^*(E, D) = \sup_{x \in E} d\,(x, D),$$

where E and D are the subsets of a given Banach space. If $h^*(E, D) < r$, then the inclusion $E \subset D + rB_*$ is verified, where B_* is the unit closed ball of a given space.

Theorem 2. *Let $F\colon X \rightsquigarrow Y$ be a set valued map, $(x, y) \in X \times Y$, $p \in X \setminus \{0\}$, $G \subset Y$ be a compact set. Assume that the inclusion*

$$G \subset \frac{\partial^L F(x, y)}{\partial p} \tag{3.1}$$

is satisfied. Then the equality

$$\lim_{\delta \to 0} \frac{1}{\delta} h^* \left(y + \delta G, F(x + \delta p)\right) = 0 \tag{3.2}$$

is verified.

P r o o f. If $\dfrac{\partial^L F(x, y)}{\partial p} = \emptyset$, the theorem is true trivially.

Suppose that $\dfrac{\partial^L F(x, y)}{\partial p} \neq \emptyset$ and let us assume the contrary, i. e., let the equality (3.2) is not satisfied. Then there exist a sequence $\{\delta_i\}_{i=1}^{\infty}$ and $\alpha_* > 0$ such that $\delta_i \to 0^+$ as $i \to +\infty$ and

$$\lim_{i \to \infty} \frac{1}{\delta_i} h^* \left(y + \delta_i G, F(x + \delta_i p)\right) = \alpha_*. \tag{3.3}$$

Let $\sigma_* < \alpha_*$ be an arbitrary number. It follows from (3.3) that there exists $N_1 > 0$ such that

$$h^* \left(y + \delta_i G, F(x + \delta_i p)\right) > \frac{\sigma_*}{2} \delta_i$$

for every $i > N_1$. On behalf of definition of the Hausdorff deviation, we have that for each $i > N_1$ there exists a $g_i \in G$ such that the inequality

$$d\left(y + \delta_i g_i, F(x + \delta_i p)\right) > \frac{\sigma_*}{4} \delta_i \tag{3.4}$$

holds. Since $G \subset Y$ is a compact set, $g_i \in G$ for every $i = 1, 2, \ldots$, then without loss of generality it is possible to assume that $g_i \to g_*$ as $i \to \infty$ and $g_* \in G$.

According to (3.1) we have $g_* \in \dfrac{\partial^L F(x, y)}{\partial p}$, and therefore

$$\lim_{\delta \to 0} \frac{1}{\delta} d\left(y + \delta g_*, F(x + \delta p)\right) = 0. \tag{3.5}$$

Since $g_i \to g_*$ as $i \to \infty$, then it follows from (3.5) that

$$\begin{aligned} \lim_{i \to \infty} \frac{1}{\delta_i} d\left(y + \delta_i g_i, F(x + \delta_i p)\right) &\leq \lim_{i \to \infty} \frac{1}{\delta_i} \big[d\left(y + \delta_i g_i, y + \delta_i g_*\right) \\ &+ d\left(y + \delta_i g_*, F(x + \delta_i p)\right) \big] = \lim_{i \to \infty} \|g_i - g_*\| = 0. \end{aligned} \tag{3.6}$$

The relations (3.4) and (3.6) contradict. Proof is completed. □

Corollary 1. *Suppose that the conditions of Theorem 2 are satisfied. Then there exist* $\delta_* > 0$ *and a function* $r(\cdot)\colon (0, \delta_*] \to [0, \infty)$ *such that* $r(\delta) \to 0^+$ *as* $\delta \to 0^+$ *and*

$$y + \delta G \subset F(x + \delta p) + \delta r(\delta) B_Y$$

for every $\delta \in (0, \delta_*]$ *where* $B_Y = \{y \in Y \colon \|y\| \le 1\}$.

Theorem 2 is not true if in (3.1) the lower directional derivative set will be replaced by the upper directional derivative set.

Example 4. Let the set-valued map $F(\cdot)\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined as in Example 3 by (2.1), $p = 1$, $(x, y) = (0, 0) \in grF(\cdot)$. One can show that $\dfrac{\partial^U F(0,0)}{\partial 1} = [-1, 1]$. Let $G = [-1, 1]$. Then $G \subset \dfrac{\partial^U F(0,0)}{\partial 1}$, but

$$\liminf_{\delta \to 0^+} \frac{1}{\delta} h^* \left(0 + \delta G, F(0 + \delta \cdot 1)\right) = \liminf_{\delta \to 0^+} \frac{1}{\delta} h^* \left(0 + \delta \cdot [-1, 1], \delta \cdot \sin \frac{1}{\delta}\right)$$
$$= \liminf_{\delta \to 0^+} h^* \left([-1, 1], \sin \frac{1}{\delta}\right) \ge \frac{1}{2},$$

and equality (3.2) is not held.

Note, that in Theorem 2 the lower directional derivative set $\dfrac{\partial^L F(x,y)}{\partial p}$ can not be replaced by the set $D^L F(x, y)|(p)$.

Example 5. Let the set-valued map $F(\cdot)\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined as in Example 3 by (2.1), $p = 1$, $(x, y) = (0, 0) \in grF(\cdot)$. According to Example 3, $0 \in D^L F(0, 0)|(1)$. Let $G = \{0\}$. Then $G \subset D^L F(0, 0)|(1)$.

Since

$$\frac{1}{\delta} h^* \left(0 + \delta G, F(0 + \delta \cdot 1)\right) = \frac{1}{\delta} h^* \left(0, \delta \cdot \sin \frac{1}{\delta}\right) = \frac{1}{\delta} \left|\delta \cdot \sin \frac{1}{\delta}\right| = \left|\sin \frac{1}{\delta}\right|$$

for every $\delta > 0$, then the limit $\lim\limits_{\delta \to 0+} \dfrac{1}{\delta} h^* \left(0 + \delta G, F(0 + \delta \cdot 1)\right)$ does not exist, which verifies that equality (3.2) is not satisfied.

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МАТЕМАТИКА 2022. Т. 32. Вып. 3. С. 403–414.

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Верхние и нижние производные множества по направлениям и дифференциалы многозначных отображений

Ключевые слова: многозначное отображение, контингентный конус, дифференциал, производное множество по направлениям, отклонение Хаусдорфа.

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В данной работе изучаются производные множества по направлениям и дифференциалы заданного многозначного отображения. Указаны различные соотношения между производными множествами по направлениям и дифференциалами многозначного отображения. Установлено, что каждое компактное подмножество множества нижних производных может быть использовано для нижней аппроксимации заданного многозначного отображения. Вычисляются и сравниваются верхние и нижние контингентные конусы некоторых множеств на плоскости.

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