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SUFFICIENT TURING INSTABILITY CONDITIONS FOR THE SCHNAKENBERG SYSTEM

A classical reaction–diffusion system, the Schnakenberg system, is under consideration in a bounded domain $\Omega \subset \mathbb{R}^m$ with Neumann boundary conditions. We study diffusion-driven instability of a stationary spatially homogeneous solution of this system, also called the Turing instability, which arises when the diffusion coefficient d changes. An analytical description of the region of necessary and sufficient conditions for the Turing instability in the parameter plane is obtained by analyzing the linearized system in diffusionless and diffusion approximations. It is shown that one of the boundaries of the region of necessary conditions is an envelope of the family of curves that bound the region of sufficient conditions. Moreover, the intersection points of two consecutive curves of this family lie on a straight line whose slope depends on the eigenvalues of the Laplace operator and does not depend on the diffusion coefficient. We find an analytical expression for the critical diffusion coefficient at which the stability of the equilibrium position of the system is lost. We derive conditions under which the set of wavenumbers corresponding to neutral stability modes is countable, finite, or empty. It is shown that the semiaxis $d > 1$ can be represented as a countable union of half-intervals with split points expressed in terms of the eigenvalues of the Laplace operator; each half-interval is characterized by the minimum wavenumber of loss of stability.

Keywords: reaction–diffusion systems, Schnakenberg system, Turing space, critical wavenumber.

Introduction

Systems of semi-linear parabolic equations, also called reaction–diffusion systems, find numerous applications in modeling of various physical, biological, and other processes [1]. Bifurcations of stationary solutions of such systems, as a result of which spatial-temporal structures are formed, have been considered by many authors [2–13].

A special role in the processes of structure formation and self-organization is played by the mechanism of diffusion instability, discovered by A. Turing [11] and reflected in modern research [2–4, 7–10]. In the classical monography of J. Murray [12], the Turing instability is used, in particular, for modeling the color of animal skins.

The Schnackenberg system [13] was proposed to describe a two-component chemical reaction and has the form

$$\frac{du}{dt} = u^2 v - u + a, \qquad \frac{dv}{dt} = -u^2 v + b, \tag{0.1}$$

where $u(t), v(t)$ are time-dependent concentrations of chemicals; a, b are constant concentrations that are assumed to be given.

Let us take into account the diffusion process, which leads to concentration equalization. Suppose that the concentrations of the interacting substances $u = u(x, t), v = v(x, t)$ depend not only on time t, but also on a spatial variable x, and x changes in a bounded domain $\Omega \subset \mathbb{R}^m$ for $m = 1, 2, 3$. We will assume that for $m = 2, 3$ the boundary $\partial\Omega$ of the domain is sufficiently smooth, $\partial\Omega \in C^2$, or Ω is a rectangular parallelepiped.

Let $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \ldots + \dfrac{\partial^2}{\partial x_m^2}$ be the Laplace operator and ψ_k be the eigenfunctions corresponding to the eigenvalues μ_k of the operator $-\Delta$ under Neumann boundary conditions, $k = 0, 1, 2, \ldots$.

$$\Delta\psi_k + \mu_k\psi_k = 0, \quad x \in \Omega, \qquad \left.\frac{\partial\psi_k}{\partial n}\right|_{\partial\Omega} = 0. \tag{0.2}$$

In this paper it is assumed that all eigenvalues μ_k (0.2) are simple. In one-dimensional case, this condition is fulfilled automatically; in the case of a rectangular parallelepiped, it is sufficient to assume that the squares of its sides are incommensurable.

Let D_1, D_2 be the diffusion coefficients for the first and the second substances, respectively. Then from (0.1) we arrive at the spatially distributed Schnackenberg system:

$$u_t = D_1\Delta u + f(u, v), \qquad v_t = D_2\Delta v + g(u, v), \tag{0.3}$$

where

$$f(u, v) = u^2 v - u + a, \qquad g(u, v) = -u^2 v + b. \tag{0.4}$$

After change of variables $x_i \to \sqrt{D_1}x_i$, $i = 1, 2, \ldots, m$, and introduction of a new notation for the diffusion coefficient $d = \dfrac{D_1}{D_2}$, the system (0.3) takes the form

$$u_t = \Delta u + f(u, v), \qquad v_t = d\Delta v + g(u, v). \tag{0.5}$$

We require Neumann boundary conditions on $\partial\Omega$:

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0, \tag{0.6}$$

where n is the outward normal to the boundary. In what follows, we will call the system (0.5) with boundary conditions (0.6), in which the terms of the reaction f and g are given by (0.4), the Schnackenberg system with diffusion.

By standard techniques, the system (0.4)–(0.6) is reduced to an ordinary differential equation in the Hilbert space H of vector functions $\boldsymbol{w} = (u, v)$ with components $u, v \in L_2(\Omega)$. Let an operator $A_0 \colon H \to H$ be defined on the set of vector functions $\boldsymbol{w} = (u, v)$, whose components belong to the Sobolev space $W_2^2(\Omega)$ and satisfy the boundary conditions (0.6), and acts according to the rule $A_0 = -D\Delta$, where D is a matrix of diffusion coefficients

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}. \tag{0.7}$$

Hence, the system (0.4)–(0.6) takes the form

$$\boldsymbol{w}_t = A_0\boldsymbol{w} + K(\boldsymbol{w}), \qquad K(\boldsymbol{w}) = (f, g); \qquad \boldsymbol{w} \in H. \tag{0.8}$$

The equation (0.8) is usually considered with some initial conditions. If the initial conditions are spatially homogeneous, that is, do not depend on x, then the solution will be spatially homogeneous. Thus, $\mathbb{R}^2$ is an invariant subspace of the diffusion system for any diffusion coefficient.

In particular, the equilibrium position of the system (0.1),

$$(u_0, v_0) = \left(a + b, \ \frac{b}{(a + b)^2} \right) \tag{0.9}$$

is a spatially homogeneous solution to the diffusion system (0.8). According to the chemical meaning of the system, we require the positivity of the solution (u_0, v_0)

$$a + b > 0, \qquad b > 0. \tag{0.10}$$

The Turing instability, or diffusion-driven instability, of a spatially homogeneous solution (u_0, v_0) of a diffusive system is characterized by the property of stability in the absence of diffusion and instability in the presence of diffusion. Starting with Turing's work [11], diffusion-driven instability is established by analyzing linearized systems without diffusion and with diffusion.

Denote by J the Jacoby matrix of ordinary differential equations system (0.1) at the point (u_0, v_0) (0.9):

$$\text{J} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}\bigg|_{(u_0, v_0)}. \tag{0.11}$$

Here

$$\begin{aligned} f_u &= \frac{b - a}{a + b}, & f_v &= (a + b)^2, \\ g_u &= -\frac{2b}{a + b}, & g_v &= -(a + b)^2. \end{aligned} \tag{0.12}$$

Then the system (0.1) linearized in a neighborhood of (u_0, v_0) has the form

$$\frac{d\boldsymbol{y}}{dt} = \text{J}\boldsymbol{y}, \qquad \boldsymbol{y} \in \mathbb{R}^2. \tag{0.13}$$

Now consider the Schnackenberg system with diffusion. We linearize (0.8) in a neighborhood of the equilibrium position (u_0, v_0):

$$\boldsymbol{w}_t = \text{A}\boldsymbol{w}, \qquad \text{A} = \text{A}_0 + \text{J}, \tag{0.14}$$

where the operator A is defined on the domain of the operator A_0. It is known that the spectrum of the operator A is discrete because its resolvent is a compact operator in H [14].

Definition 1. An equilibrium (u_0, v_0) of a system with diffusion (0.8) is called *Turing unstable* if two conditions are satisfied. First, the eigenvalues of the diffusion-free system (0.13) linearized in the vicinity of the equilibrium state lie strictly in the left half-plane of the complex plane. Second, there is an eigenvalue of the linearized diffusion system (0.14) lying in the right half-plane.

Note that, strictly speaking, being interested in the bifurcations of the main solution, it is necessary to justify the linearization for both the system in the diffusionless approximation and the system in the presence of diffusion.

If the first condition of the Turing instability is satisfied, then the equilibrium position of the nonlinear system (0.1) is exponentially asymptotically stable in a finite-dimensional phase space. This implies exponential asymptotic stability of the steady state (u_0, v_0) in the invariant subspace $\mathbb{R}^2$ of the infinite-dimensional phase space H of the system with diffusion (0.8).

The justification of linearization in the stability problem for a wide class of parabolic systems was substantiated in [15]. We will not dwell on checking the conditions of these theorems in this paper.

The system (0.4)–(0.6) contains parameters a, b, satisfying conditions (0.10), and the diffusion coefficient d. For practical applications, it is important to find the Turing instability region as well as the critical value of the diffusion parameter.

Definition 2. A region in the parameter plane (a, b) containing those parameters for which the Turing instability takes place and the diffusion coefficient d is fixed is called the *Turing instability region.*

Now let the parameters a and b be fixed, and the diffusion coefficient d be changed. Then the eigenvalues of the linearized system (0.14) might be considered as functions of the parameter d. It will be shown below that for d less than a certain value, all eigenvalues of the system (0.14) lie strictly in the left half-plane of the complex plane.

We are interested in a critical case, when the eigenvalues cross the imaginary axis. In general case, this is possible when either the imaginary axis is crossed by a pair of purely imaginary eigenvalues (then an oscillatory loss of stability occurs), or the eigenvalue passes through zero (which corresponds to a monotonic loss of stability). It is known that the Turing instability corresponds to a monotonic loss of stability [11, 12].

Nevertheless, a large number of works are devoted to an oscillatory loss of stability in reaction–diffusion systems, as a result of which the Hopf bifurcation arises [2–8]. The emergence of spatially inhomogeneous regimes in reaction–diffusion systems as a result of the oscillatory or monotonic loss of stability of the stationary solution of the system has been studied by many authors. For instance, [3, 4, 7, 8] follow the approach given in [2] for diffusive predator–prey system. In [5] critical values of a control parameter, corresponding to the oscillatory and monotonic loss of stability are found for a two-component reaction–diffusion Rayleigh system under Dirichlet and mixed boundary conditions. In [6] spatially-inhomogeneous auto-oscillations and stationary regimes are found which are stable in infinite–dimensional invariant subspaces of the Rayleigh system under Neumann boundary condition. In [7] Turing–Hopf bifurcation as a result of diffusion and time delay is investigated.

In [8] bifurcations of an equilibrium, such as Turing, steady state and Hopf bifurcations, have been studied for Schnakenberg-type system. In particular, sufficient conditions for the Turing instability are given.

In [9], for a three-variable system of reaction–diffusion equations, it is shown that the interaction of two types of diffusive instabilities, namely Turing and wave instabilities, leads to formation not only pure stationary and autowave structures, but also more complicated mixed regimes. In present work, as in [7], the role of the bifurcation parameter is played by the ratio of the diffusion coefficients $d = \dfrac{D_1}{D_2}$.

Definition 3. *A critical value of the parameter* d *is a value of* d_c *such that the spectrum of the linearized problem* (0.14) *lies strictly in the left half-plane of the complex plane, except for the eigenvalue* $\lambda(d_c) = 0$, *and the intersection of the imaginary axis occurs transversely:*

$$\lambda'|_{d=d_c} \neq 0, \tag{0.15}$$

where prime ′ means differentiation with respect to the parameter d.

The aim of this paper is to find an explicit analytical representation of the boundary of the Turing instability region for the Schnackenberg system with diffusion, as well as the critical value of the diffusion coefficient, and to determine the range of Turing instability wavenumbers so that it depends on the diffusion coefficient. Some preliminary results are formulated in [9].

Let us dwell briefly on the differences between the present work and the works [4, 7], which are the closest in the subject, in the part in which they interact with our work. In [4], bifurcations of the steady state solutions of the Schnackenberg system in a neighbourhood of the equilibrium position are studied, the statements are proved in one-dimensional case $x \in (0, \ell\pi)$. In [4], the

monotonic loss of stability as a whole is under consideration, the parameter a plays the role of a bifurcation parameter, while the parameter b and the diffusion coefficient d remain fixed. In contrast, in the present work, the results do not depend on the dimension of the spatial domain. As in [4], we consider the case of one-dimensional expanding domains, but only as a special case. We also use the substitution of variables proposed in [4] to simplify the calculations. This will be discussed in the next section.

The results of [7] refer, in particular, to the Schnackenberg system (0.3) in one-dimensional case $x \in (0, 1)$ with the boundary conditions (0.6) under the assumption that the diffusion coefficients are given as $D_1 = \varepsilon d, D_2 = d$. For this system an explicit expression is obtained for one of the boundaries of the region of sufficient conditions for Turing instability on the parameter plane (d, ε) for fixed values of the parameters (a, b) and the wavenumber $k \in N$.

In the present paper, the region of necessary and sufficient conditions for the Turing instability on the plane of the initial parameters (a, b) is explicitly found. These conditions are found for an arbitrary domain Ω and are expressed in terms of the eigenvalues of the Laplace operator with Neumann boundary conditions.

For any value of the diffusion coefficient satisfying the condition $d > 1$, the range of wavenumbers k at which the Turing instability occurs is indicated. In the particular case of positive values of the parameters a and b, it is shown that the range of wavenumbers is finite, and the values of this range are explicitly found.

§1. Necessary conditions for Turing instability

In this section, based on well-known results on the necessary conditions for the Turing instability [12], we introduce definitions that will be convenient in what follows.

Let us find the trace and the determinant of the matrix J (0.11)–(0.12) and write down the conditions under which the eigenvalues of the system without diffusion (0.13) lie strictly in the left half-plane of the complex plane:

$$\text{Tr}(\text{J}) \equiv \frac{b - a - (a + b)^3}{a + b} < 0, \qquad \text{Det}(\text{J}) \equiv (a + b)^2 > 0. \tag{1.1}$$

From the assumption (0.10) it follows that the condition $\text{Det}(\text{J}) > 0$ is always satisfied. Therefore, the stability conditions (1.1) for the diffusionless approximation take the form:

$$b - a < (a + b)^3. \tag{1.2}$$

Let's introduce new variables:

$$Y = b - a, \qquad X = a + b. \tag{1.3}$$

Hence, taking into account (0.10), the condition (1.2) takes the form:

$$X > 0, \qquad Y < X^3. \tag{1.4}$$

The convenience of change of variables (1.3) is fairly obvious. These variables are also used in [4]. We introduced it independently of this paper.

Now we consider a linear spectral problem for the operator A (0.14) in H:

$$\text{A}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}, \qquad \boldsymbol{\varphi} \neq 0. \tag{1.5}$$

Let us derive the necessary conditions under which the operator A has an eigenvalue in the right half-plane.

It is known [16] that the operator A has a matrix representation in the basis consisting of the eigenvectors $\{e_1\psi_k, e_2\psi_k\}_{k=0}^{+\infty}$ of the operator A_0 (here $e_1 = (1,0), e_2 = (0,1)$; ψ_k are the eigenfunctions (0.2)). The matrix of the operator A has a block-diagonal form with the matrices $J_k = J - \mu_k D$ on the diagonal, where D is defined in (0.7):

$$
J_k = \begin{pmatrix} \dfrac{b-a}{a+b} - \mu_k & (a+b)^2 \\ -\dfrac{2b}{a+b} & -(a+b)^2 - d\mu_k \end{pmatrix}.
\tag{1.6}
$$

Looking for an eigenvector φ in the form of eigenfunctions series

$$
\varphi = \sum_{k=0}^{+\infty} C_k \psi_k, \qquad C_k = (c_k^1, c_k^2),
\tag{1.7}
$$

after substitution the series (1.7) into (1.5) and equating the coefficients of the same eigenfunctions, for each fixed k we obtain a linear system with the matrix J_k (1.6), which corresponds to the eigenvalue λ_k and the eigenvector C_k:

$$
J_k C_k = \lambda_k C_k, \qquad C_k \neq 0.
$$

Being interested in the instability of the equilibrium position (u_0, v_0), we find the trace and the determinant of the matrix J_k:

$$
\mathrm{Tr}(J_k) = \mathrm{Tr}(J) - (1+d)\mu_k,
$$
$$
\mathrm{Det}(J_k) = d\mu_k^2 + \left((a+b)^2 - d \cdot \frac{b-a}{a+b}\right)\mu_k + \mathrm{Det}(J).
\tag{1.8}
$$

Since $\mathrm{Det}(J) = (a+b)^2 > 0$ due to (0.10), the loss of stability can occur only for $k > 0$. In this case $\mu_k > 0$. Therefore, the inequality $\mathrm{Tr}(J_k) < \mathrm{Tr}(J) < 0$ is fulfilled, and the equilibrium position (u_0, v_0) can become unstable only if $\mathrm{Det}(J_k) = 0$. In particular, it follows that only a monotonic loss of stability corresponds to diffusion driven instability.

By $h(\mu)$ we denote the polynomial

$$
h(\mu) \equiv d\mu^2 + \left((a+b)^2 - d \cdot \frac{b-a}{a+b}\right)\mu + (a+b)^2,
\tag{1.9}
$$

where $h(\mu_k) = \mathrm{Det}(J_k)$ (1.8). After change of variables (1.3), $h(\mu)$ takes the form:

$$
h(\mu) \equiv d\mu^2 + \left(X^2 - d \cdot \frac{Y}{X}\right)\mu + X^2.
\tag{1.10}
$$

If the condition $d < \dfrac{X^3}{Y}$ is satisfied, then all the coefficients of the trinomial $h(\mu)$ (1.10) are positive, and, therefore, its roots lie strictly in the left half-plane. For $d = \dfrac{X^3}{Y}$, the trinomial $h(\mu)$ has a pair of purely imaginary roots. Being interested only in the case of real and positive roots of the polynomial $h(\mu)$, we require the second coefficient to be negative

$$
d > \frac{X^3}{Y},
\tag{1.11}
$$

and the discriminant of $h(\mu)$ to be non-negative:

$$\left(dY - X^3\right)^2 \geqslant 4dX^4. \tag{1.12}$$

Taken together, the conditions (1.11) and (1.12) lead to the inequality:

$$dY - X^3 \geqslant 2\sqrt{d}X^2. \tag{1.13}$$

Taking into account the inequalities (1.4), which are satisfied in the diffusionless approximation, and (1.13), the necessary conditions for the Turing instability in the variables (X, Y) have the form:

$$X > 0, \qquad Y < X^3, \qquad Y \geqslant \frac{1}{d}X^3 + \frac{2}{\sqrt{d}}X^2. \tag{1.14}$$

Note that (1.14) implies a restriction on Y (and parameters a and b)

$$Y = b - a > 0,$$

and also on the diffusion coefficient d

$$d > 1.$$

Now we present the definitions that we introduced to describe the boundary of the Turing instability region.

Definition 4. *Zero trace curve* is the curve $Y = X^3$ in the plane (X, Y) corresponding to vanishing of the trace of the matrix J.

Definition 5. The curve $Y_0 = Y_0(X)$ in the plane (X, Y) corresponding to vanishing of the discriminant of the trinomial $h(\mu)$ will be called a *discriminant curve:*

$$Y_0 = \frac{1}{d}X^3 + \frac{2}{\sqrt{d}}X^2. \tag{1.15}$$

Thus, in the half-plane $X > 0$, the region of necessary conditions for the Turing instability is bounded by zero trace curve and the discriminant curve. Using the change of variables (1.3), in a similar way one can define zero trace curve and the discriminant curve in the plane of the original parameters (a, b) of the Schnackenberg system.

Obviously, zero trace curve and the discriminant curve intersect at $X = 0, Y = 0$. Let's find the intersection point $N_* = (X_*, X_*^3)$ of these curves with positive coordinates. The abscissa of this point is found by the formula

$$X_* = \frac{2\sqrt{d}}{d - 1}. \tag{1.16}$$

It is easy to verify that for $X > X_*$ zero trace curve lies above the discriminant curve in the (X, Y)-plane and, therefore, the region of necessary conditions for the Turing instability (1.14) is not empty. The formula (1.16) has not been encountered in the literature yet.

Let us find the values of the diffusion coefficient at which the point with coordinates (X, Y) lies not below the discriminant curve. To do this, solve the inequality $Y \geqslant Y_0$ with respect to parameter d. We have: $d \geqslant d_0$, where d_0 is the value of d corresponding to the case when the point (X, Y) lies on the discriminant curve. We find d_0:

$$\sqrt{d_0} = \frac{X^2 + \sqrt{X^3(X + Y)}}{Y}. \tag{1.17}$$

Considering the trinomial $h(\mu)$ (1.9) as a function of not only the parameter μ, but also the diffusion coefficient d, we conclude that for $d = d_0$ the vertex of the parabola $y = h(\mu)$ crosses the μ-axis. In [12], the value of d_0 (1.17) was defined in the variables a, b, and this value of d_0 was called the critical diffusion coefficient. In addition, the region in the plane of parameters (a, b), in which the necessary conditions for Turing instability are satisfied, was called Turing space.

We emphasize that for $d \geqslant d_0$ the necessary conditions for the Turing instability are satisfied, but, generally speaking, the sufficient conditions are not satisfied.

If the polynomial h (1.9) depended not on a discrete set of eigenvalues μ_k, but on a continuous nonnegative parameter μ, then the necessary condition (1.14) would be sufficient. This follows from the fact that the spectrum of the operator $-\Delta$ defined in the whole space ($\mathbb{R}^1, \mathbb{R}^2$ or $\mathbb{R}^3$) fills the non-negative semiaxis. The discreteness of the spectrum $\{\mu_k\}_{k=1}^{\infty}$ for bounded domains leads to the fact that one of the boundaries of the domain of sufficient conditions does not coincide with the discriminant curve. Indeed, for $d = d_0$, the vertex of the parabola $h(\mu)$ is touching the μ-axis, but the abscissa of the vertex should not coincide with some eigenvalue μ_k.

The discreteness of the spectrum of the Laplace operator for bounded domains has been taking into account in the literature to describe the region of the Turing instability [4, 7]. Nevertheless, no explicit formulas of the boundary of the sufficient conditions region have been obtained.

§ 2. Sufficient conditions for Turing instability

The sufficient conditions for the Turing instability are the necessary conditions supplemented by the condition for the existence of such a value of $k \in N$ for which

$$\mathrm{Det}(\mathrm{J_k}) = h(\mu_k) \leqslant 0.$$

Definition 6. Let the parameters a and b be fixed, and the diffusion coefficient take on a critical value $d = d_c$. A *critical wavenumber* is a value of $k \in N$ for which the eigenvalue of the Laplace operator μ_k is a root of the trinomial $h(\mu)$: $h(\mu_k) = 0$.

Let us denote by d_k such value of d for which μ_k coincides with one of the roots of the quadratic trinomial $h(\mu_k)$. Find d_k by equating the expression for $h(\mu_k)$ to zero:

$$d_k = \frac{X^3}{Y - \mu_k X} \cdot \frac{\mu_k + 1}{\mu_k}. \tag{2.1}$$

Since d_k is positive, it follows a restriction on the system parameters:

$$\mu_1 \leqslant \mu_k < \frac{Y}{X}. \tag{2.2}$$

We denote by $Y = Y_k(X)$ the curves corresponding to the coincidence of the values of μ_k with the roots of the quadratic trinomial $h(\mu_k)$ for different k. The case when the point (X, Y) belongs to one of these curves corresponds to the equality $d = d_k$. Solving (2.1) for $Y_k(X)$, we obtain the equation of the required curves:

$$Y_k(X) = \frac{\mu_k + 1}{\mu_k \cdot d} \cdot X^3 + \mu_k X. \tag{2.3}$$

Definition 7. Let $F(X, Y, k) = 0$ be a one-parameter family of curves in the half-plane $X > 0$ depending on a natural parameter k. A smooth curve Φ is called an *envelope* of this family if each curve of the family has a common point with the curve Φ, and at this point the curve of the family has a common tangent line with Φ.

Using the explicit expressions for discriminant curve $Y_0(X)$ (1.15) and the curves $Y_k(X)$, $k \geqslant 1$ (2.3), it is easy to prove the statement.

Theorem 1 (about discriminant curve). *The discriminant curve $Y_0 = Y_0(X)$ has the following properties:*

(a) *it lies no higher than one-parameter family of curves $Y_k(X)$ in the half-plane $X > 0$;*

(b) *it has with each curve $Y_k(X)$ a unique common point $T_k = (X_{k,0}, Y_{k,0})$, whose coordinates are found by the formulas*

$$X_{k,0} = \mu_k \sqrt{d}, \qquad Y_{k,0} = \mu_k^2(\mu_k + 2)\sqrt{d}; \qquad (2.4)$$

(c) *it is an envelope of one-parameter family of the curves $Y_k(X)$;*

(d) *the curves $Y_k(X)$ and the curve $Y_0(X)$ are convex downward for any $X > 0$.*

From (2.4) it follows that the common points of the discriminant curve and the curve $Y_k(X)$ for fixed k are located on a straight line with the slope $\gamma_{k,0} = \mu_k(\mu_k + 2)$ which does not depend on the diffusion coefficient d. It will be shown below that an analogous property holds for the intersection points of the curves $Y_k(X)$.

Our goal is to find a region in which not only necessary, but also sufficient conditions for the Turing instability are satisfied. Further, it will be shown that one of the boundaries of this region consists of fragments of the curves $Y_k(X)$, and the other boundary coincides with zero trace curve. To describe a mutual arrangement of the curves $Y_k(X)$, we use the following statement.

Theorem 2 (about $Y_k(X)$ and $Y_m(X)$). *Let $1 \leqslant k < m$. Then*

(a) *in the half-plane $X > 0$ the curves $Y_k(X)$ and $Y_m(X)$ have one intersection point $(X_{k,m}, Y_{k,m})$, and its coordinates are found by the formulas:*

$$X_{k,m} = \sqrt{\mu_k \mu_m d}, \qquad Y_{k,m} = \gamma_{k,m} X_{k,m}, \qquad \gamma_{k,m} = \mu_k + \mu_m + \mu_k \mu_m;$$

(b) *for $X < X_{k,m}$, the curve $Y_k(X)$ is located below the curve $Y_m(X)$:*

$$Y_k(X) < Y_m(X);$$

for $X > X_{k,m}$„ the curve $Y_k(X)$ is located above the curve $Y_m(X)$.

P r o o f. Both statements follow from the inequality $Y_k(X) \leqslant Y_m(X)$ considered in the half-plane $X > 0$. This inequality takes the form:

$$(\mu_k - \mu_m)\left(\frac{X^2}{d\mu_k\mu_m} - 1\right) \geqslant 0.$$

Taking into account the assumption that the eigenvalues μ_k are simple and $\mu_k < \mu_m$ for $1 \leqslant k < m$, we obtain the required statement. $\square$

For convenience, we introduce the following notation. By γ_0 we denote the first nonzero eigenvalue of the operator $-\Delta$, by $X_{0,m}$ we denote the abscissa of the intersection point N of the curve $Y_m(X)$ with zero trace curve $Y = X^3$:

$$\gamma_0 = \mu_1, \qquad X_{0,m}^2 = \frac{X_{m,0}^2}{\mu_m(d-1) - 1}, \qquad (2.5)$$

where $X_{m,0}$ is defined in (2.4); the intersection point $N = (X_{0,m}, X_{0,m}^3)$. Theorem 2 implies the properties of a mutual arrangement of two consecutive curves $Y_k(X)$ and $Y_{k+1}(X)$.

Theorem 3 (about $Y_k(X)$ and $Y_{k+1}(X)$). *The curves $Y_k(X)$, $k \geqslant 1$, have the following properties:*

(a) *in the half-plane $X > 0$ the curves $Y_k(X)$ and $Y_{k+1}(X)$ have a unique intersection point $C_k = (X_{k,k+1}, Y_{k,k+1})$, whose coordinates are found by the formulas*

$$X_{k,k+1} = \sqrt{\mu_k \mu_{k+1} d}, \qquad Y_{k,k+1} = \gamma_k X_{k,k+1}, \tag{2.6}$$

$$\gamma_k = \mu_k + \mu_{k+1} + \mu_k \mu_{k+1}; \tag{2.7}$$

(b) *common points $C_k = (X_{k,k+1}, Y_{k,k+1})$ of the curves $Y_k(X)$ and $Y_{k+1}(X)$ are located on a straight line with slope γ_k, which does not depend on the diffusion coefficient;*

(c) *for $X \in [X_{k-1,k}, X_{k,k+1}]$, the curve $Y_k(X)$ is located in the sector bounded by straight lines with slopes γ_{k-1} and γ_k (2.7):*

$$\gamma_{k-1} X \leqslant Y_k(X) \leqslant \gamma_k X; \tag{2.8}$$

(d) *for $X \in [X_{k-1,k}, X_{k,k+1}]$, the curve $Y_k(X)$ is located below all other curves $Y_m(X)$, $m \neq k$:*

$$Y_k(X) = \min_m Y_m(X).$$

For any fixed diffusion coefficient d, one can indicate the set of the critical wavenumbers k corresponding to the neutral stability modes ψ_k, and describe the structure of the Turing instability region. It will be shown below that under conditions (0.10), the set of wavenumbers corresponding to neutral modes is countable. If we additionally require the positiveness of the parameter a, then this set will be finite or empty.

By $Z_1(X)$ we denote the union of the curves $Y_k(X)$ for $X \in [X_{k-1,k}, X_{k,k+1}]$, $k \geqslant 1$,

$$Z_1(X) = \bigcup_{k \geqslant 1} \{Y_k(X), X \in [X_{k-1,k}, X_{k,k+1}]\}, \tag{2.9}$$

here $X_{0,1}$ is defined in (2.5) at $m = 1$ and $X_{k,k+1}$ is found in (2.6). Let N_k be an intersection point of the straight line $Y = \gamma_k X$ and zero trace curve $Y = X^3$: $N_k = (\sqrt{\gamma_k}, (\sqrt{\gamma_k})^3)$.

Theorem 4 (about Turing region for large d). *Let the following inequality hold*

$$d > 1 + \frac{1}{\mu_1} + \frac{1}{\mu_2}. \tag{2.10}$$

Then the region of necessary and sufficient conditions for the Turing instability in the (X, Y)-plane is given by the inequalities

$$X > 0, \qquad Y < X^3, \qquad Y \leqslant Z_1(X),$$

where $Z_1(X)$ is defined in (2.9), and it consists of a curved triangle NN_1C_1 bounded by the curves NN_1, NC_1 and the straight line N_1C_1:

$$
\begin{aligned}
Y &= X^3, & X_{0,1} &\leqslant X \leqslant \sqrt{\gamma_1}; \\
Y &= Y_1, & X_{0,1} &\leqslant X \leqslant X_{1,2}; \\
Y &= \gamma_1 X, & \sqrt{\gamma_1} &\leqslant X \leqslant X_{1,2};
\end{aligned}
\tag{2.11}
$$

and a union of curved quadrangles $C_{k-1} N_{k-1} N_k C_k$, $k \geqslant 2$, bounded by the curves $N_{k-1} N_k$, $C_{k-1} C_k$, and the straight lines $N_{k-1} C_{k-1}$, $N_k C_k$:

$$
\begin{aligned}
Y = X^3, \quad & \sqrt{\gamma_{k-1}} \leqslant X \leqslant \sqrt{\gamma_k}; & Y = Y_k, \quad & X_{k-1,k} \leqslant X \leqslant X_{k,k+1}; \\
Y = \gamma_{k-1} X, \quad & \sqrt{\gamma_{k-1}} \leqslant X \leqslant X_{k-1,k}; & Y = \gamma_k X, \quad & \sqrt{\gamma_k} \leqslant X \leqslant X_{k,k+1}.
\end{aligned}
\tag{2.12}
$$

P r o o f. Note that the condition (2.10) is equivalent to the following inequality

$$d\mu_1\mu_2 > \mu_1 + \mu_2 + \mu_1\mu_2,$$

or, which is the same, the inequality

$$X_{1,2}^2 > \gamma_1, \tag{2.13}$$

which means that the abscissa of the intersection point C_1 of the curves Y_1 and Y_2 is greater than the abscissa of the intersection point N_1 of zero trace curve and the straight line $Y = \gamma_1 X$: $X_{1,2} > \sqrt{\gamma_1}$. At the same time, the inequality (2.13) implies the inequality

$$X_{2,3}^2 > \gamma_2. \tag{2.14}$$

Indeed, the inequality (2.14) is equivalent to the following inequality $d > 1 + \dfrac{1}{\mu_2} + \dfrac{1}{\mu_3}$. Since $\mu_3 > \mu_1$, it follows from the assumption (2.10). Thus, the abscissa of C_2 is greater than the abscissa of N_2. We conclude that the quadrangle $C_1N_1N_2C_2$ (2.12) for $k = 2$ belongs to the Turing instability region.

Similarly, the inequality (2.14) implies the inequality $X_{3,4}^2 > \gamma_3$. Repeating the previous reasoning, we find that all quadrangles $C_{k-1}N_{k-1}N_kC_k$ (2.12) belong to the Turing instability region for $k \geqslant 2$.

The region of instability will also include the triangle NN_1C_1 (2.11), the two vertices of which, N_1 and C_1, as shown in (2.13), are different. The third vertex N is an intersection of zero trace curve and Y_1. Indeed, it is easy to show that the following inequality holds:

$$X_{0,1}^2 < \gamma_1. \tag{2.15}$$

Then from (2.14) and (2.15) it follows that the intersection point N of zero trace curve and Y_1 is located to the left of the intersection point C_1 of the curves Y_1 and Y_2: $X_{0,1}^2 < X_{1,2}^2$. $\qquad\square$

Thus, if the diffusion coefficient satisfies the inequality (2.10), then the Turing instability region is bounded by zero trace curve and fragments of the curves $Y_k(X)$, starting from $k = 1$. Similarly, if the following inequality holds

$$1 + \frac{1}{\mu_2} + \frac{1}{\mu_3} < d \leqslant 1 + \frac{1}{\mu_1} + \frac{1}{\mu_2}, \tag{2.16}$$

then the Turing instability region is bounded by zero trace curve and fragments of the curves $Y_k(X)$, starting from $k = 2$. Indeed, it follows from the condition (2.16) that the sign is reversed in the inequalities (2.10) and (2.14). Therefore, the quadrangles (2.12) belong to the Turing instability region, starting from $k = 3$. Similarly, in the case of (2.16), the abscissa $X_{0,1}$ of the vertex N of the triangle (2.11) is replaced by $X_{0,2}$.

For all $m \geqslant 2$ we define

$$Z_m(X) = \bigcup_{k \geqslant m+1} \{Y_k(X), X \in [X_{k-1,k}, X_{k,k+1}]\} \bigcup \{Y_m(X), X \in [X_{0,m}, X_{m,m+1}]\}. \tag{2.17}$$

The next statement is proved similarly to Theorem 4, taking into account the remark above.

Theorem 5 (about Turing region). *Let $m \geqslant 2$ be fixed and the following inequality hold*

$$1 + \frac{1}{\mu_m} + \frac{1}{\mu_{m+1}} < d \leqslant 1 + \frac{1}{\mu_{m-1}} + \frac{1}{\mu_m}. \tag{2.18}$$

Then the region of necessary and sufficient conditions for the Turing instability in the (X, Y)-plane is given by the inequalities

$$X > 0, \qquad Y < X^3, \qquad Y \geqslant Z_m(X),$$

where $Z_m(X)$ is defined in (2.17), and it consists of a curved triangle NN_mC_m bounded by the curves NN_m, NC_m and the straight line N_mC_m:

$$
\begin{aligned}
Y &= X^3, & X_{0,m} &\leqslant X \leqslant \sqrt{\gamma_m}; \\
Y &= Y_m, & X_{0,m} &\leqslant X \leqslant X_{m,m+1}; \\
Y &= \gamma_m X, & \sqrt{\gamma_m} &\leqslant X \leqslant X_{m,m+1};
\end{aligned}
\tag{2.19}
$$

and a union of curvilinear quadrangles $C_{k-1}N_{k-1}N_kC_k$ (2.12) for $k \geqslant m+1$.

If the conditions (0.10) are strengthened by the assumption that the parameter a is positive

$$a > 0, \qquad b > 0, \tag{2.20}$$

then the range of wavenumbers corresponding to neutral modes becomes finite. Note that the positiveness of a is equivalent to the following inequality in the variables X and Y:

$$Y < X. \tag{2.21}$$

Let $A = (1, 1)$ and $B_k = (\widetilde{X}_k, \widetilde{X}_k)$ be the intersection points of $Y = X$ with zero trace curve $Y = X^3$ and $Y = Y_k$ respectively. Here

$$\widetilde{X}_k^2 = \frac{1 - \mu_k}{1 + \mu_k} \mu_k d, \qquad \mu_k < 1.$$

From (2.2) and (2.21) it follows that under the conditions of Theorem 4 for $\gamma_0 = \mu_1 \geqslant 1$ the Turing instability does not arise. Similarly, under the conditions of Theorem 5, (2.2) and (2.21) imply that for $\gamma_{m-1} \geqslant 1$ the Turing instability region is empty. In this case, the range of critical wavenumbers is determined by the condition $\gamma_{k-1} < 1$. Thus, under the assumption (2.20), from Theorems 4 and 5, we deduce the following statements.

Theorem 6 (about Turing region for positive a and large d). *Suppose that the diffusion coefficient satisfies the inequality (2.10), and the parameters a and b satisfy the conditions (2.20), then the following statements are true:*

(a) *if $\gamma_0 \geqslant 1$, then the set of critical wavenumbers k is empty and the Turing instability does not arise;*

(b) *if $\gamma_0 < 1$, but $\gamma_1 \geqslant 1$, then the Turing instability region consists of a curvilinear triangle NAB_1 bounded by the curves NA, NB_1 and the straight line AB_1:*

$$Y = X^3, \quad X_{0,1} \leqslant X \leqslant 1; \qquad Y = Y_1, \quad X_{0,1} \leqslant X \leqslant \tilde{X}_1; \qquad Y = X, \quad 1 \leqslant X \leqslant \tilde{X}_1;$$

(c) *if $\gamma_1 < 1$, but $\gamma_2 \geqslant 1$, then the Turing instability region consists of a curvilinear triangle NN_1C_1 (2.11) and a curvilinear quadrilateral $C_1N_1AB_2$ bounded by the curves N_1A, C_1B_2 and the straight lines N_1C_1, AB_2:*

$$
\begin{aligned}
Y &= X^3, & \sqrt{\gamma_1} &\leqslant X \leqslant 1; & Y &= Y_2, & X_{1,2} &\leqslant X \leqslant \tilde{X}_2; \\
Y &= \gamma_1 X, & \sqrt{\gamma_1} &\leqslant X \leqslant X_{1,2}; & Y &= X, & 1 &\leqslant X \leqslant \tilde{X}_2;
\end{aligned}
$$

(d) *if $\gamma_k < 1$ for any $k \in [0, M-1]$, where $M \geqslant 3$, but $\gamma_M \geqslant 1$, then the Turing instability region consists of a curvilinear triangle NN_1C_1 (2.11), a union of curvilinear quadrangles $C_{k-1}N_{k-1}N_kC_k$ (2.12) for $k \in [2, M-1]$ and a curvilinear quadrilateral $C_{M-1}N_{M-1}AB_M$ bounded by the curves $N_{M-1}A, C_{M-1}B_M$ and the straight lines $N_{M-1}C_{M-1}, AB_M$.*

Theorem 7 (about Turing region for positive a). *Let $m \geqslant 2$ be fixed and the inequality (2.18) hold, the parameters a and b satisfy the condition (2.20), then the following statements are true:*

(a) *if $\gamma_{m-1} \geqslant 1$, then the set of critical wavenumbers k is empty and the Turing instability does not arise;*

(b) *if $\gamma_{m-1} < 1$, but $\gamma_m \geqslant 1$, then the Turing instability region consists of a curvilinear triangle NAB_m bounded by the curves NA, NB_m and the straight line AB_m:*

$$Y = X^3, \quad X_{0,m} \leqslant X \leqslant 1; \qquad Y = Y_m, \quad X_{0,m} \leqslant X \leqslant \tilde{X}_m; \qquad Y = X, \quad 1 \leqslant X \leqslant \tilde{X}_m;$$

(c) *if $\gamma_m < 1$, but $\gamma_{m+1} \geqslant 1$, then the Turing instability region consists of a curvilinear triangle NN_mC_m (2.19) and a curvilinear quadrilateral $C_mN_mAB_{m+1}$ bounded by the curves N_mA, C_mB_{m+1} and the straight lines N_mC_m, AB_{m+1}:*

$$Y = X^3, \quad \sqrt{\gamma_m} \leqslant X \leqslant 1; \qquad Y = Y_{m+1}, \quad X_{m,m+1} \leqslant X \leqslant \tilde{X}_{m+1};$$

$$Y = \gamma_m X, \quad \sqrt{\gamma_m} \leqslant X \leqslant X_{m,m+1}; \qquad Y = X, \quad 1 \leqslant X \leqslant \tilde{X}_{m+1};$$

(d) *if $\gamma_k < 1$ for any $k \in [m-1, M-1]$, where $M \geqslant m+2$, but $\gamma_M \geqslant 1$, then the Turing instability region consists of a curvilinear triangle NN_mC_m (2.19), a union of curvilinear quadrangles $C_{k-1}N_{k-1}N_kC_k$ (2.12) for $k \in [m, M-1]$ and a curvilinear quadrilateral $C_{M-1}N_{M-1}AB$ bounded by the curves $N_{M-1}A, C_{M-1}B_M$ and the straight lines $N_{M-1}C_{M-1}, AB_M$.*

Under the conditions of Theorems 4, 5, 6, 7, we find the critical value of the diffusion coefficient.

Theorem 8 (about critical diffusion coefficient). *The critical value of the diffusion coefficient for $X \in [X_{k-1,k}, X_{k,k+1}], Y = Y_k$ is determined by the equality*

$$d_c = d_k,$$

where d_k is found by the formula (2.1).

P r o o f. It follows from the properties of the curves $Y_k(X)$ (2.8) that for $X \in [X_{k-1,k}, X_{k,k+1}]$ the inequality (2.2) is satisfied and the diffusion coefficient d_k is positive. Moreover, for $d < d_c$, the stability spectrum of the linearized problem (0.14) lies strictly in the left half-plane. For $d = d_c$, the determinant $\mathrm{Det}(J_k)$ (1.8) becomes zero, which corresponds to the appearance of a zero eigenvalue $\lambda(d_c) = 0$. It remains to prove that the transversality condition (0.15) is satisfied.

The eigenvalue $\lambda(d)$ of the linearized problem (0.14), which vanishes for $d = d_c$, is found by the formula:

$$\lambda(d) = \frac{1}{2}\left(\mathrm{Tr}(J_k) + \sqrt{\mathrm{Tr}^2(J_k) - 4\,\mathrm{Det}(J_k)}\right).$$

Since $\mathrm{Tr}'(J_k) = -\mu_k$, $\mathrm{Tr}(J_k) < 0$, then

$$2\lambda'(d_c) = \mathrm{Tr}''(J) + \frac{\mathrm{Tr}(J_k) \cdot \mathrm{Tr}'(J) - 2\,\mathrm{Det}'(J_k)}{|\mathrm{Tr}(J)|} = -\frac{2\,\mathrm{Det}'(J_k)}{|\mathrm{Tr}(J)|}.$$
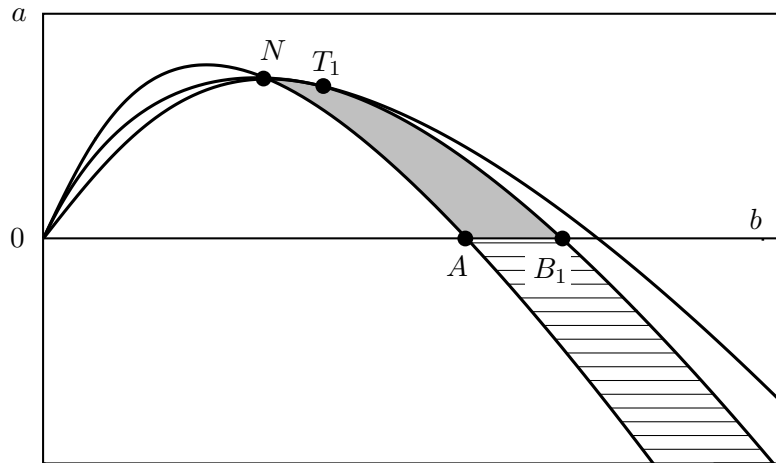
Fig. 1. Turing instability region NAB_1 for $d = 10$ and $\ell = 2\pi$
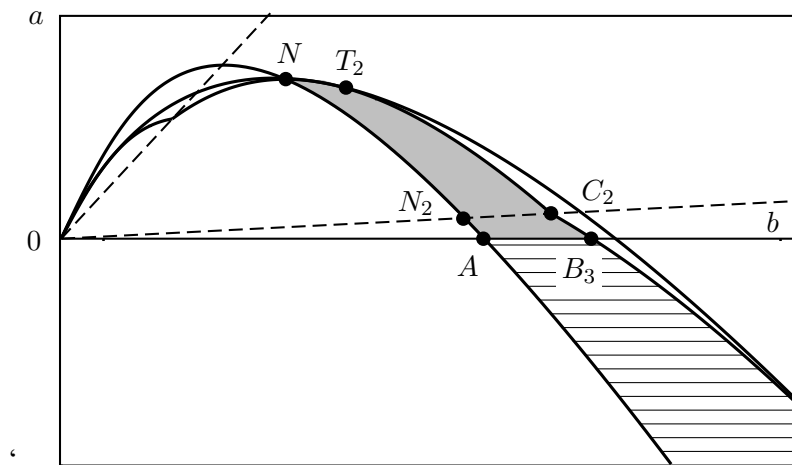

Fig. 2. Turing instability region NAB_3 for $d = 10$ and $\ell = 4\pi$
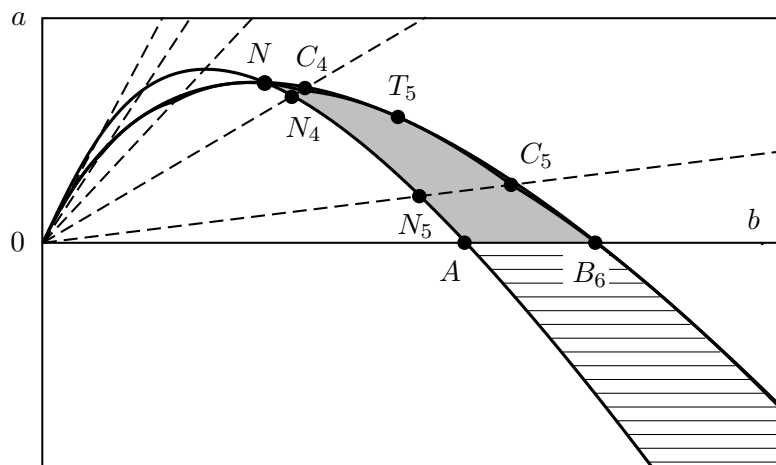

Fig. 3. Turing instability region NAB_6 for $d = 10$ and $\ell = 9\pi$

From the equality

$$\mathrm{Det}'(\mathrm{J_k}) = \mu_k^2 - \frac{Y}{X}\mu_k = \mu_k\left(\mu_k - \frac{Y}{X}\right)$$

and the conditions (2.2), it follows that the derivative $\mathrm{Det}'(\mathrm{J_k}) < 0$, and, therefore, $\lambda'(d_c) > 0$. Thus, the transversality condition is satisfied.   □

Example 1. In one-dimensional case, when $\Omega = (0, \ell)$, substituting $\mu_k = \left(\frac{\pi k}{\ell}\right)^2$ into (2.3), we get the expression for the curves $Y_k(X)$

$$Y_k(X) = \frac{\pi^2 k^2 + \ell^2}{\pi^2 k^2 \cdot d} \cdot X^3 + \left(\frac{\pi k}{l}\right)^2 \cdot X.$$

The inequality $\gamma_0 \geqslant 1$ is equivalent to the condition $\ell \leqslant \pi$. In this condition the Turing instability does not arise.

Putting $d = 10$ and assuming that conditions (2.20) are satisfied, we will consider three cases: $\ell = 2\pi$, $\ell = 4\pi$ and $\ell = 9\pi$.

In the first case, condition (b) of Theorem 6 is satisfied. Consequently, the critical value of the wavenumber $k = 1$ and the region of the Turing instability has the form of a curvilinear triangle NAB_1.

In the second case, condition (c) of Theorem 7 is satisfied for $m = 2$. Consequently, the range of wavenumbers is $k \in [2, 3]$, and the Turing instability region consists of a curvilinear triangle NN_2C_2 and a curvilinear quadrangle $C_2N_2AB_3$.

In the third case, condition (d) of Theorem 7 is satisfied for $m = 4$. Consequently, the range of wavenumbers is $k \in [4, 6]$, and the Turing instability region consists of a curvilinear triangle NN_4C_4 and two curvilinear quadrangles $C_4N_4N_5C_5$, and $C_5N_5AB_6$.

Figures 1, 2, 3 show the Turing instability region for the examples under consideration in the plane (a, b) of original parameters.

Using the matrix representation (1.6) of the operator A (0.14), it is easy to prove the following statement.

Theorem 9 (about the simplicity of zero eigenvalue). *Suppose that the diffusion coefficient satisfies the inequality (2.10) (or the inequality (2.18)), and the parameters a and b are such that the point (X, Y) belongs to the boundary of the Turing instability region $Z_1(X)$ (or $Z_m(X)$, respectively). Then the operator A has a zero eigenvalue. This eigenvalue is simple for $X \in \in (X_{k-1,k}, X_{k,k+1})$ as $k \geqslant 1$ (or $k \geqslant m$, respectively), and corresponding eigenvector has a form*

$$\varphi_k = g_k\psi_k(x); \qquad g_k = \left(X^2, \mu_k - \frac{Y}{X}\right), \tag{2.22}$$

where ψ_k is an eigenfunction of the Laplace operator found by the formula (0.2), and for $X = X_{k,k+1}$ as $k \geqslant 1$ (or $k \geqslant m$, respectively) this zero eigenvalue has the multiplicity 2 with two eigenvectors φ_k and φ_{k+1}.

A similar result on the simplicity of the eigenvalue of the linearized problem is formulated in [7], but, as we mentioned above, the dependence on other parameters, namely the diffusion coefficients, is considered.

Based on the linearization method [15], we present a scheme for analyzing a nonlinear equation. The nonlinear perturbation equation has the form

$$\boldsymbol{w}_t = \mathrm{A}\boldsymbol{w} + \mathrm{K}\boldsymbol{w}, \qquad \boldsymbol{w} = (w_1, w_2) \in H,$$

here the linear operator A is defined in (0.14), and the nonlinear operator K is represented as a sum of quadratic and cubic terms

$$\mathrm{K}\boldsymbol{w} = \mathrm{K}_1\boldsymbol{w} + \mathrm{K}_2\boldsymbol{w},$$
$$\mathrm{K}_1\boldsymbol{w} = (v_0 w_1^2 + 2u_0 w_1 w_2, -v_0 w_1^2 - 2u_0 w_1 w_2),$$
$$\mathrm{K}_2\boldsymbol{w} = (w_1^2 w_2, -w_1^2 w_2)$$

with (u_0, v_0) (0.14). The linear operator A could be represented as a sum of the Laplacian A_0 and a bounded operator J, so it generates an analytical semigroup in the functional space H of vector-functions with $L_2(\Omega)$ components. The nonlinear operator K acting from the vector space W_2^2 to H is compact in accordance with the Sobolev embedding theorems.

In addition, as shown in Theorems 8 and 9, for the critical value of the diffusion parameter and the parameters a, b belonging to the boundary of the Turing instability region, except for the piecewise points, the operator A has a simple zero eigenvalue that intersects the imaginary axis transversally. Together, these conditions make it possible to apply the Lyapunov–Schmidt reduction or the center manifold method to find secondary spatially inhomogeneous solutions in the vicinity of the boundary of the first loss of stability. The perturbations of the basic solution (u_0, v_0) can be parameterized as

$$\boldsymbol{w}(s) = s\boldsymbol{\varphi_k} + O(s^2), \qquad s \to 0,$$

where $\boldsymbol{\varphi_k}$ is defined in (2.22). A larger value of the wavenumber k corresponds to a more complex structure of the secondary solutions arising as a result of the Turing bifurcation.

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Достаточные условия неустойчивости Тьюринга для системы Шнакенберга

Классическая система реакции–диффузии — система Шнакенберга — рассматривается в ограниченной области m-мерного пространства, на границе которой предполагаются выполненными краевые условия Неймана. Изучается диффузионная неустойчивость стационарного пространственно-однородного решения этой системы, называемая также неустойчивостью Тьюринга, возникающая при изменении коэффициента диффузии d. Путем анализа линеаризованной системы в бездиффузионном и диффузионном приближениях получено аналитическое описание области необходимых и достаточных условий неустойчивости Тьюринга на плоскости параметров системы. Показано, что одна из границ области необходимых условий является огибающей семейства кривых, ограничивающих область достаточных условий. При этом точки пересечения двух соседних кривых лежат на прямой, угловой коэффициент которой зависит от собственных значений оператора Лапласа в рассматриваемой области и не зависит от коэффициента диффузии. Найдено аналитическое выражение критического коэффициента диффузии, при котором происходит потеря устойчивости положения равновесия системы. Указаны условия, в зависимости от которых множество волновых чисел, соответствующих нейтральным модам устойчивости, счетно, конечно или пусто. Показано, что полуось $d > 1$ можно представить в виде счетного объединения полуинтервалов, каждому из которых соответствует минимальное волновое число, при котором происходит потеря устойчивости, причем точки разбиения полуоси выражаются через собственные значения оператора Лапласа в рассматриваемой области.

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