MSC: 30C45

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ON A SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY LINEAR OPERATOR

The present paper introduces and studies the subclass $A_n(m, \beta, p, q, \lambda)$ of univalent functions with negative coefficients defined by new linear operator J^{λ} in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. The main task is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems. Neighborhood and radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $A_n(m, \beta, p, q, \lambda)$ are studied.

Keywords: analytic univalent function, Hadamard product, Ruscheweyh derivative, distortion theorems, closure theorems.

§ 1. Introduction

Let $A(n)$ denote the class of functions normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N = 1, 2, 3, \ldots), \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $A^-(n)$ be the subclass of $A(n)$, consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N, \quad a_k \geqslant 0). \tag{1.2}$$

For functions $f(z) \in A(n)$ given by (1.1) and $g(z) \in A(n)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \quad (n \in N),$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$f(z) * g(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k \quad (z \in \mathcal{U}). \tag{1.3}$$

We define the function $\Phi(a, c; z)$ by

$$\Phi(a, c; z) = z + \sum_{k=n+1}^{\infty} \frac{(a)_{k-1}}{(b)_{k-1}} z^k \quad (a \in R, \quad c \in R - \{0, -1, -2, \ldots\}),$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & \text{if} \quad k = 0, \\ a(a+1)(a+2) \cdot \cdots \cdot (a+k-1), & \text{if} \quad k \in N. \end{cases}$$

We consider a functions $\Phi^-(a, c; z)$ defined by the convolution

$$\Phi(a, c; z) * \Phi^-(a, c; z) = \frac{z}{(1-z)^{\lambda+1}},$$

where $\lambda > -1, z \in \mathcal{U}$. This function yields the following family of linear operators

$$I^{\lambda}(a,c;z)f(z) = \Phi^{-}(a,c;z) * f(z), \quad z \in \mathcal{U},$$

where $a, c \in -\{0,-1,-2,\ldots\}$. For a function $f \in A^{-}(n)$, it follows that for $\lambda > -1$

$$I^{\lambda}(a,c;z)f(z) = z - \sum_{k=n+1}^{\infty} \frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} a_k z^k$$

is the Cho–Kown–Srivastava integral operator [11]. Further, denote by $D^{\lambda} : A(n) \to A(n)$ the Ruscheweyh derivative of order λ defined by [8, 13, 14].

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = \frac{z(z^{\lambda-1}f(z))^{\lambda}}{\lambda!} \qquad (f(z) \in A(n)),$$

where $\lambda > -1$. Now, for $f \in A^{-}(n)$ we defined a new operator [4]

$$J^{\lambda} f(z) = I^{\lambda}(a,c;z)f(z) * D^{\lambda} f(z) = z - \sum_{k=n+1}^{\infty} \frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}} B_k(\lambda) a_k z^k, \tag{1.4}$$

where $B_k(\lambda) = \dfrac{\Gamma(\lambda+k)}{\Gamma(\lambda+1)(k-1)!}$, $\lambda > -1$, $z \in \mathcal{U}$. Next, we define the following

Definition 1. Let the function $f(z)$ be of the form (1.2). Then $f(z)$ is said to be *in the class* $A_n(m, \beta, p, q, \lambda)$ if it satisfies the following criterion

$$\left| \frac{\frac{z(J^{\lambda}f(z))'}{J^{\lambda}f(z)} - 1}{2p\left(\frac{z(J^{\lambda}f(z))'}{J^{\lambda}f(z)} - q\right) - m\left(\frac{z(J^{\lambda}f(z))'}{J^{\lambda}f(z)} - 1\right)} \right| < \beta, \tag{1.5}$$

where $|z| < 1$, $0 < \beta \leqslant 1$, $\frac{1}{2} \leqslant p \leqslant 1$, $0 \leqslant q \leqslant \frac{1}{2}p$, and $\frac{1}{2} < m \leqslant 1$.

Normalized, univalent analytic functions have been extensively studied by (for example) Aghalary and Kulkarni [1], Aouf [2], Khairnar and Meena More [6], Juma and Kulkarni [5].

Remark 1. We note that by specializing the parameters m, β, p, q and λ we have the following subclasses.
(i) If $f(z) \in A_n(m, \beta, p, q, \lambda)$ with $a = c = 1$ and $\lambda = 0$ in (1.4), then $J^{\lambda} f(z) = f(z)$.
(ii) The class $A_n(1, 1, 1, 0, \lambda)$ is the class of starlike functions in $\mathcal{U}$.
(iii) The class $A_n(1, 1, 1, q, \lambda)$ is the class of starlike functions of order q $(0 \leqslant q < 1)$.
(iv) The class $A_n(1, \beta, \frac{(\alpha+1)}{2}, 0, \lambda)$ is the class studied by Lakshminarasimhan [9].
(v) The class $A_n(1, \beta, p, q, \lambda)$ is the class studied by S. R Kulkarni [7].

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $A_n(m, \beta, p, q, \lambda)$. Then we obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ $(0 \leqslant \leqslant \delta < 1)$ for this functions.

§ 2. Coefficient inequality

Theorem 1. *Let the function $f(z)$ be of the form* (1.2). *Then $f(z)$ is in the class $A_n(m, \beta, p, q, \lambda)$ if and only if*

$$\sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1} B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_k \leqslant 2p\beta(1-q), \tag{2.1}$$

where $0 < \beta \leqslant 1$, $\frac{1}{2} \leqslant p \leqslant 1$, $0 \leqslant q \leqslant \frac{1}{2}p$, $\frac{1}{2} < m \leqslant 1$, $a, c \in R - \{0,-1,-2,\ldots\}$, $z \in \mathcal{U}$.

P r o o f. Suppose that (2.1) holds true. Then we find that

$$\left|\frac{\frac{z(J^\lambda f(z))'}{J^\lambda f(z)}-1}{2p(\frac{z(J^\lambda f(z))'}{J^\lambda f(z)}-q)-m(\frac{z(J^\lambda f(z))'}{J^\lambda f(z)}-1)}\right|=$$

$$=\left|\frac{-\sum_{k=n+1}^{\infty}(k-1)\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_kz^k}{2p(1-q)z+\sum_{k=n+1}^{\infty}[m(k-1)-2p(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_kz^k}\right|\leqslant$$

$$\leqslant\frac{\sum_{k=n+1}^{\infty}(k-1)\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_k|z|^{k-1}}{2p(1-q)-\sum_{k=n+1}^{\infty}[m(k-1)-2p(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_k|z|^{k-1}}<\beta.$$

Choosing values of z on real axis and letting $z\to 1^-$, we have

$$[(k-1)(1-m\beta)-2p\beta(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_k-2p\beta(1-q)\leqslant 0.$$

By hypothesis, thus by maximum modulus theorem, we have

$$f(z)\in A_n(m,\beta,p,q,\lambda).$$

Conversely, assume that $f(z)\in A_n(m,\beta,p,q,\lambda)$, then from the definition of $f(z)$, we have

$$\left|\frac{\frac{z(J^\lambda f(z))'}{J^\lambda f(z)}-1}{2p(\frac{z(J^\lambda f(z))'}{J^\lambda f(z)}-q)-m(\frac{z(J^\lambda f(z))'}{J^\lambda f(z)}-1)}\right|<\beta,$$

that is

$$\left|\frac{-\sum_{k=n+1}^{\infty}(k-1)\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_kz^k}{2p(1-q)z+\sum_{k=n+1}^{\infty}[m(k-1)-2p(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_kz^k}\right|<\beta.$$

By the fact $|\mathrm{Re}\,(z)|\leqslant|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{k=n+1}^{\infty}(k-1)\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_k|z|^{k-1}}{2p(1-q)z+\sum_{k=n+1}^{\infty}[m(k-1)-2p(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_k|z|^{k-1}}\right\}<\beta.$$

We choose the values of z on real axis such that $\frac{(J^\lambda f(z))'}{J^\lambda f(z)}$ is real and upon clearing, the denominator of the above expression and allowing $z\to 1^-$ through real values, we obtain

$$\sum_{k=n+1}^{\infty}[(k-1)(1-m\beta)-2p\beta(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}}a_k\leqslant 2p\beta(1-q).$$ □

Corollary 1. *Let the function $f(z)$ of the form* (1.2) *be in the class $A_n(m,\beta,p,q,\lambda)$. Then*

$$a_k\leqslant\frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}\quad(k\geqslant n+1,\ n\in N),$$

where $0<\beta\leqslant 1$, $\frac{1}{2}\leqslant p\leqslant 1$, $0\leqslant q\leqslant\frac{1}{2}p$, $\frac{1}{2}<m\leqslant 1$, $a,c\in R-\{0,-1,-2,\ldots\}$, $z\in\mathcal{U}$.

Remark 2. If $f(z)\in A_n(1,\beta,p,q,\lambda)$, then

$$a_k\leqslant\frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}\quad(k\geqslant n+1,\ n\in N)$$

and equality holds for

$$f(z)=z-\frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}z^k.$$

Remark 3. If $f(z) \in A_n(1,\beta,1,q,\lambda)$, then

$$a_k \leqslant \frac{2\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta)-2\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \quad (k \geqslant n+1,\ n \in N)$$

and equality holds for

$$f(z) = z - \frac{2\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-\beta)-2\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k.$$

Remark 4. If $f(z) \in A_n(1,1,1,q,\lambda)$, then

$$a_k \leqslant \frac{-(1-q)(1)_{k-1}(a)_{k-1}}{(k-q)(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \quad (k \geqslant n+1, n \in N)$$

and equality holds for

$$f(z) = z + \frac{(1-q)(1)_{k-1}(a)_{k-1}}{(k-q)(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k.$$

§ 3. Distortion theorems

Theorem 2. *Let the function $f(z)$ of the form (1.2) be in the class $A_n(m,\beta,p,q,\lambda)$. Then for $|z| = r < 1$, we have*

$$r - \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)} r^{n+1} \leqslant |f(z)| \leqslant$$

$$\leqslant r + \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)} r^{n+1}.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)} z^{n+1}.$$

P r o o f. Suppose that $f(z) \in A_n(m,\beta,p,q,\lambda)$. By the inequality (2.1), since

$$[n(1-m\beta)-2p\beta(n+1-q)]\frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n}B_{n+1}(\lambda)$$

is non decreasing and positive for $k \geqslant n+1$, we have

$$[n(1-m\beta)-2p\beta(n+1-q)]\frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n}B_{n+1}(\lambda)\sum_{k=n+1}^{\infty} a_k \leqslant$$

$$\leqslant \sum_{k=n+1}^{\infty}[(k-1)(1-m\beta)-2p\beta(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}}B_k(\lambda)a_k \leqslant 2p\beta(1-q).$$

This is equivalent to

$$\sum_{k=n+1}^{\infty} a_k \leqslant \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}. \tag{3.1}$$

Using (1.2) and (3.1), we obtain

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k,$$

$$|f(z)| \geqslant |z| - \sum_{k=n+1}^{\infty} a_k |z|^k \geqslant r - \sum_{k=n+1}^{\infty} a_k r^k \geqslant r - r^{n+1} \sum_{k=n+1}^{\infty} a_k \geqslant$$

$$\geqslant r - \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^{n+1}.$$

Similarly

$$|f(z)| \leqslant r + \frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^{n+1}.$$

This completes the proof of Theorem 2. □

Theorem 3. *Let the function $f(z)$ of the form (1.2) be in the class $A_n(m, \beta, p, q, \lambda)$. Then for $|z| = r < 1$, we have*

$$1 - \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n \leqslant |f'(z)| \leqslant$$

$$\leqslant 1 + \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} z^{n+1}.$$

P r o o f. From (1.2) and (3.1) we have $f'(z) = 1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}$,

$$|f'(z)| \geqslant 1 - \sum_{k=n+1}^{\infty} k a_k |z|^{k-1} \geqslant 1 - \sum_{k=n+1}^{\infty} a_k r^{k-1} \geqslant 1 - r^n \sum_{k=n+1}^{\infty} k a_k \geqslant$$

$$\geqslant 1 - \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n.$$

Similarly

$$|f(z)| \leqslant 1 + \frac{2p\beta(1+n)(1-q)(1)_n(a)_n}{[n(1-m\beta) - 2p\beta(n+1-q)](c)_n(\lambda+1)_n B_{n+1}(\lambda)} r^n.$$

This completes the proof of Theorem 3. □

§ 4. Closure theorem

Theorem 4. *Let the functions f_j $(j = 1, 2, \ldots, s)$ defined by*

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geqslant 0) \tag{4.1}$$

are in the class $A_n(m, \beta, p, q, \lambda)$. Then the function $h(z)$ defined by $h(z) = \sum_{j=1}^{s} \mu_j f_j(z) \quad (\mu_j \geqslant 0)$ is also in the class $A_n(m, \beta, p, q, \lambda)$ if $\sum_{j=1}^{s} \mu_j = 1$.

P r o o f. We can write

$$h(z) = \sum_{j=1}^{s} \mu_j (z - \sum_{k=n+1}^{\infty} a_{k,j} z^k) = \sum_{j=1}^{s} \mu_j z - \sum_{j=1}^{s} \sum_{k=n+1}^{\infty} \mu_j a_{k,j} z^k = z - \sum_{k=n+1}^{\infty} \sum_{j=1}^{s} \mu_j a_{k,j} z^k.$$

Furthermore, since the functions $f_j(z)$ $(j = 1, 2, \ldots, s)$ are in the class $A_n(m, \beta, p, q, \lambda)$, then from Theorem 1 we have

$$\sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_{k,j} \leqslant 2p\beta(1-q).$$

Thus it is enough to prove that

$$\sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} (\sum_{j=1}^{s} \mu_j a_{k,j}) =$$

$$= \sum_{j=1}^{s} \mu_j \sum_{k=n+1}^{\infty} [(k-1)(1-m\beta) - 2p\beta(k-q)] \frac{(c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{(1)_{k-1}(a)_{k-1}} a_{k,j} \leqslant$$

$$\leqslant \sum_{j=1}^{s} \mu_j 2p\beta(1-q) = 2p\beta(1-q).$$

Hence the proof is complete. □

Corollary 2. *Let the functions* f_j $(j = 1, 2)$ *defined by* (4.1) *are in the class* $A_n(m, \beta, p, q, \lambda)$. *Then the function* $h(z)$ *defined by* $h(z) = (1-\zeta)f_1(z) + \zeta f_2(z)$, $0 \leqslant \zeta \leqslant 1$, *is also in the class* $A_n(m, \beta, p, q, \lambda)$.

§ 5. Extreme points

Theorem 5. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} z^k \quad (k \geqslant n+1,\ n \in N).$$

Then the function $f(z)$ *of the form* (1.2) *is in the class* $A_n(m, \beta, p, q, \lambda)$ *if and only if it can be expressed in the form*

$$f(z) = \mu_1 f_1(z) + \sum_{k=n+1}^{\infty} \mu_k f_k(z),$$

where $\mu_1 \geqslant 0$, $\mu_k \geqslant 0$, $k \geqslant n+1$, *and* $\mu_1 + \sum\limits_{k=n+1}^{\infty} \mu_k = 1$.

P r o o f. Assume that $f(z)$ can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum\limits_{k=n+1}^{\infty} \mu_k f_k(z)$,

$$f(z) = z - \sum_{k=n+1}^{\infty} \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \mu_k z^k.$$

Thus

$$\sum_{k=n+1}^{\infty} \frac{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(1)_{k-1}(a)_{k-1}} \cdot$$

$$\cdot \sum_{k=n+1}^{\infty} \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \mu_k = \sum_{k=n+1}^{\infty} \mu_k = 1 - \mu_1 \leqslant 1.$$

Hence $f(z) \in A_n(m, \beta, p, q, \lambda)$.

Conversely, assume that $f(z) \in A_n(m, \beta, p, q, \lambda)$. Setting

$$\mu_k = \frac{[(k-1)(1-m\beta) - 2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(1)_{k-1}(a)_{k-1}} a_k,$$

since $\mu_1 = 1 - \sum\limits_{k=n+1}^{\infty} \mu_k$. Thus $f(z) = \mu_1 f_1(z) + \sum\limits_{k=n+1}^{\infty} \mu_k f_k(z)$. Hence the proof is complete. □

Corollary 3. *The extreme points of the class $A_n(m,\beta,p,q,\lambda)$ are the functions $f_1(z)=z$ and*

$$f_k(z)=z-\frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}z^k \quad (k\geqslant n+1,\ n\in N).$$

§ 6. Neighborhood property

The concept of neighborhood of analytic function was first introduced by Goodman [3]. Ruscheweyh [14], Raina and Srivastava [12], Orhan and Kamali [10], Silverman [15] investigated this concept for the elements of several famous subclass of analytic functions. We define the (n,δ)-neighborhood of a function $f(z)\in A^-(n)$ by

$$N_{n,\delta}(f)=\{g\in A^-(n): g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k \text{ and } \sum_{k=n+1}^{\infty}k|a_k-b_k|\leqslant\delta\}. \tag{6.1}$$

In particular, for $e(z)=z$

$$N_{n,\delta}(e)=\{g\in A^-(n): g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k \text{ and } \sum_{k=n+1}^{\infty}k|b_k|\leqslant\delta\}. \tag{6.2}$$

Furthermore, a function of the form (1.2) is said to be in the class $A_n^{\sigma}(m,\beta,p,q,\lambda)$ if there exists a function $h(z)\in A_n(m,\beta,p,q,\lambda)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right|<1-\sigma \quad (z\in\mathcal{U},\ 0\leqslant\sigma<1). \tag{6.3}$$

Theorem 6. *If*

$$\frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}}B_k(\lambda)\geqslant\frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n}B_{n+1}(\lambda) \quad (k\geqslant n+1,\ n\in N) \tag{6.4}$$

and

$$\delta=\frac{2p\beta(1-q)(1)_n(a)_n}{[(n)(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)},$$

then $A_n(m,\beta,p,q,\lambda)\in N_{n,\delta}(e)$.

P r o o f. Let $f(z)\in A_n(m,\beta,p,q,\lambda)$. Then in view of assertion (2.1) of Theorem 1 and the condition (6.4), we get

$$[n(1-m\beta)-2p\beta(n+1-q)]\frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n}B_{n+1}(\lambda)\sum_{k=n+1}^{\infty}a_k\leqslant$$

$$\leqslant\sum_{k=n+1}^{\infty}[(k-1)(1-m\beta)-2p\beta(k-q)]\frac{(c)_{k-1}(\lambda+1)_{k-1}}{(1)_{k-1}(a)_{k-1}}B_k(\lambda)a_k\leqslant 2p\beta(1-q),$$

which implies

$$\sum_{k=n+1}^{\infty}a_k\leqslant\frac{2p\beta(1-q)(1)_n(a)_n}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}. \tag{6.5}$$

Appling assertion (2.1) of Theorem 1 in conjunction with (6.5), we obtain

$$[n(1-m\beta)-2p\beta(n+1-q)]\frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n}B_{n+1}(\lambda)\sum_{k=n+1}^{\infty}a_k\leqslant 2p\beta(1-q),$$

$$(n+1)[n(1-m\beta)-2p\beta(n+1-q)]\frac{(c)_n(\lambda+1)_n}{(1)_n(a)_n}B_{n+1}(\lambda)\sum_{k=n+1}^{\infty}a_k\leqslant 2p\beta(n+1)(1-q),$$

$$\sum_{k=n+1}^{\infty}ka_k\leqslant\frac{2p\beta(n+1)(1-q)(1)_n(a)_n}{[(n)(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}=\delta.$$

By virtue of (6.1), we have $f(z)\in N_{n,\delta}(e)$. This completes the proof of Theorem 6. □

Theorem 7. *If* $h(z)\in A_n(m,\beta,p,q,\lambda)$ *and*

$$\sigma=1-\frac{\delta}{(n+1)}\frac{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)-2p\beta(1-q)(1)_n(a)_n},\qquad(6.6)$$

then

$$N_{n,\delta}(h)\subset A_n^{\sigma}(m,\beta,p,q,\lambda).$$

P r o o f. Suppose that $f\in N_{n,\delta}(h)$, we then find from (1.2) that

$$\sum_{k=n+1}^{\infty}k|a_k-b_k|\leqslant\delta,$$

which readily implies the following coefficient inequality

$$\sum_{k=n+1}^{\infty}|a_k-b_k|\leqslant\frac{\delta}{n+1}\quad(n\in N).\qquad(6.7)$$

Next, since $h\in A_n(m,\beta,p,q,\lambda)$ in the view of (6.5), we have

$$\sum_{k=n+1}^{\infty}b_k\leqslant\frac{2p\beta(1-q)(1)_n(a)_n}{[(n)(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}.\qquad(6.8)$$

Using (6.7) and (6.8), we get

$$\left|\frac{f(z)}{h(z)}-1\right|\leqslant\frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}\leqslant\frac{\delta}{(n+1)\left(1-\frac{2p\beta(1-q)(1)_n(a)_n}{[(n)(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}\right)}\leqslant$$

$$\leqslant\frac{\delta}{(n+1)}\left(\frac{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)}{[n(1-m\beta)-2p\beta(n+1-q)](c)_n(\lambda+1)_nB_{n+1}(\lambda)-2p\beta(1-q)(1)_n(a)_n}\right)=1-\sigma,$$

provided that σ is given by (6.6), thus by condition (6.3), $f\in A^{\sigma}(m,\beta,p,q,\lambda)$. □

§ 7. Radii of starlikeness, convexity and close-to-convexity

Theorem 8. *Let the function* $f(z)$ *of the form* (1.2) *be in the class* $A_n(m,\beta,p,q,\lambda)$. *Then* f *is univalent starlike of order* δ $(0\leqslant\delta<1)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_k\left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}}\right\}^{\frac{1}{k-1}}.$$

The result is sharp for the function $f(z)$ *given by*

$$f_k(z)=z-\frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}\quad(k\geqslant n+1,\ n\in N).$$

P r o o f. It suffices to show that $\left|\dfrac{zf'(z)}{f(z)}-1\right| \leqslant 1-\delta$, $|z|<r_1$. We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{-\sum_{k=n+1}^{\infty}(k-1)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k z^{k-1}}\right| \leqslant \frac{\sum_{k=n+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k|z|^{k-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{k=n+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k|z|^{k-1}} \leqslant 1-\delta.$$

It is equivalent to

$$\sum_{k=n+1}^{\infty}(k-\delta)a_k|z|^{k-1} \leqslant 1-\delta.$$

Using Theorem 1, we obtain

$$|z| \leqslant \left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}}\right\}^{\frac{1}{k-1}}.$$

Hence the proof is complete. □

Theorem 9. *Let the function $f(z)$ of the form* (1.2) *be in the class $A_n(m,\beta,p,q,\lambda)$. Then f is univalent convex of order δ $(0\leqslant\delta<1)$ in $|z|<r_2$, where*

$$r_2 = \inf_k \left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}}\right\}^{\frac{1}{k-1}}.$$

The result is sharp for the function $f(z)$ given by

$$f_k(z) = z - \frac{2p\beta(1-q)(1)_{k-1}(a)_{k-1}}{[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)} \quad (k\geqslant n+1,\ n\in N). \tag{7.1}$$

P r o o f. It suffices to show that $\left|\frac{zf''(z)}{f'(z)}\right| \leqslant 1-\delta$, $|z|<r_2$. We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{k=n+1}^{\infty}k(k-1)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}ka_k z^{k-1}}\right| \leqslant \frac{\sum_{k=n+1}^{\infty}k(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}ka_k|z|^{k-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{k=n+1}^{\infty}k(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}ka_k|z|^{k-1}} \leqslant 1-\delta.$$

It is equivalent to

$$\sum_{k=n+1}^{\infty}k(k-\delta)a_k|z|^{k-1} \leqslant 1-\delta.$$

Using Theorem 1, we obtain

$$|z|^{k-1} \leqslant \left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}}\right\},$$

$$|z| \leqslant \left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(k-\delta)(1)_{k-1}(a)_{k-1}}\right\}^{\frac{1}{k-1}}.$$

Hence the proof is complete. □

Theorem 10. *Let the function $f(z)$ of the form* (1.2) *be in the class $A_n(m,\beta,p,q,\lambda)$. Then f is univalent close-to-convex of order δ $(0\leqslant\delta<1)$ in $|z|<r_3$, where*

$$r_3=\inf_k\left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(1)_{k-1}(a)_{k-1}}\right\}^{\frac{1}{k-1}}.$$

The result is sharp for the function $f(z)$ given by (7.1).

P r o o f. It suffices to show that $|f'(z)-1|\leqslant 1-\delta$, $|z|<r_3$. We have

$$|f'(z)-1|=|-\sum_{k=n+1}^{\infty}ka_kz^{k-1}|\leqslant\sum_{k=n+1}^{\infty}ka_k|z|^{k-1}.$$

Thus $|f'(z)-1|\leqslant 1-\delta$, if

$$\sum_{k=n+1}^{\infty}\frac{ka_k}{1-\delta}|z|^{k-1}\leqslant 1.$$

Using Theorem 1, the above inequality holds true if

$$|z|^{k-1}\leqslant\left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(1)_{k-1}(a)_{k-1}}\right\}$$

or

$$|z|\leqslant\left\{\frac{(1-\delta)[(k-1)(1-m\beta)-2p\beta(k-q)](c)_{k-1}(\lambda+1)_{k-1}B_k(\lambda)}{2p\beta k(1-q)(1)_{k-1}(a)_{k-1}}\right\}^{\frac{1}{k-1}}.$$

Hence the proof is complete. □

REFERENCES

1. Aghalary R., Kulkarni S. Some theorem on univalent functions, *J. India Acad. Math.*, 2002, vol. 24, no. 1, pp. 81–93.
2. Aouf M.K. Neighborhoods of certain class of analytic functions with negative coefficients, *International Journal of Mathematics and Mathematical Sciences*, 2006, Article ID 38258, 6 p. http://dx.doi.org/10.1155/IJMMS/2006/38258
3. Goodman A.W. Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, 1957, vol. 8, no. 3, pp. 598–601.
4. Juma A.R.S., Zirar H. Properties of a subclass of p-valent functions defined by new operator V_p^λ, *An. Univ. Oradea, Fasc. Mat.*, 2014, vol. 21, no. 1, pp. 73–82.
5. Juma A.R.S., Kulkarni S.R. On univalent functions with negative coefficients by using generalized Salagean operator, *Filomat*, 2007, vol. 21, no. 2, pp. 173–184. http://dx.doi.org/10.2298/FIL0702173J
6. Khairnar S.M., More M. Certain family of analytic and univalent functions with normalized conditions, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 2008, vol. 24, no. 3, pp. 333–344.
7. Kulkarni S.R. Some problems connected with univalent functions, *Ph.D Thesis*, 1981, Shivaji University, Kolhapur.
8. Kumar V., Shukla S.L. Multivalent functions defined by Ruscheweyh derivatives. II, *Indian J. Pure Appl. Math.*, 1984, vol. 15, pp. 1228–1238.
9. Lakshminarasimhan T.V. On subclasses of functions starlike in the unit disc, *J. Indian Math. Soc., New Ser.*, 1977, vol. 41, pp. 233–243.
10. Orhan H., Kamali M. Neighborhoods of a class of analytic functions with negative coefficients, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 2005, vol. 21, no. 1, pp. 55–61.
11. Patel J., Cho N.E., Srivastava H.M. Certain subclasses of multivalent functions associated with a family of linear operators, *Mathematical and Computer Modelling*, 2006, vol. 43, no. 3–4, pp. 320–338.
12. Raina R.K., Srivastava H.M. Inclusion and neighborhood properties of some analytic and multivalent functions, *JIPAM, J. Inequal. Pure Appl. Math.*, 2006, vol. 7, no. 1, paper no. 5, 6 p.
13. Ruscheweyh S. New criteria for univalent functions, *Proc. Amer. Math. Soc.*, 1975, vol. 49, no. 1, pp. 109–115.

14. Ruscheweyh S. Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, 1981, vol. 81, no. 4, pp. 521–527.
15. Silverman H. Univalent functions with negative coefficents, *Proc. Amer. Math. Soc.*, 1975, vol. 51, no. 1, pp. 109–116.

Received 29.04.2015

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А. Р. С. Джума, М. Ш. Абдул-Хуссейн, М. Ф. Хани

Об одном подклассе однолистных функций с отрицательными коэффициентами, заданном линейным оператором

Ключевые слова: аналитические однолистные функции, произведение Адамара, производная Рушевея, теоремы искажения, теоремы о замыкании.

УДК 517.53

В работе вводится и исследуется подкласс $A_n(m,\beta,p,q,\lambda)$ однолистных функций с отрицательными коэффициентами, определяемый новым линейным оператором J^λ в открытом единичном круге $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Основной задачей является изучение следующих свойств и характеристик: оценки коэффициентов, теоремы искажения, теоремы о замыкании, окрестность функции, радиусы звездообразности, выпуклости и почти выпуклости функций, принадлежащих классу $A_n(m,\beta,p,q,\lambda)$.

СПИСОК ЛИТЕРАТУРЫ

1. Aghalary R., Kulkarni S. Some theorem on univalent functions // J. India Acad. Math. 2002. Vol. 24. № 1. P. 81–93.
2. Aouf M.K. Neighborhoods of certain class of analytic functions with negative coefficients // International Journal of Mathematics and Mathematical Sciences. 2006. Article ID 38258. 6 p. http://dx.doi.org/10.1155/IJMMS/2006/38258
3. Goodman A.W. Univalent functions and nonanalytic curves // Proc. Amer. Math. Soc. 1957. Vol. 8. № 3. P. 598–601.
4. Juma A.R.S., Zirar H. Properties of a subclass of p-valent functions defined by new operator V_p^λ // An. Univ. Oradea. Fasc. Mat. 2014. Vol. 21. № 1. P. 73–82.
5. Juma A.R.S., Kulkarni S.R. On univalent functions with negative coefficients by using generalized Salagean operator // Filomat. 2007. Vol. 21. № 2. P. 173–184. http://dx.doi.org/10.2298/FIL0702173J
6. Khairnar S.M., More M. Certain family of analytic and univalent functions with normalized conditions // Acta Math. Acad. Paedagog. Nyházi. (N.S.). 2008. Vol. 24. № 3. P. 333–344.
7. Kulkarni S.R. Some problems connected with univalent functions: Ph.D Thesis, 1981. Shivaji University, Kolhapur.
8. Kumar V., Shukla S.L. Multivalent functions defined by Ruscheweyh derivatives. II // Indian J. Pure Appl. Math. 1984. Vol. 15. P. 1228–1238.
9. Lakshminarasimhan T.V. On subclasses of functions starlike in the unit disc // J. Indian Math. Soc. New Ser. 1977. Vol. 41. P. 233–243.
10. Orhan H., Kamali M. Neighborhoods of a class of analytic functions with negative coefficients // Acta Math. Acad. Paedagog. Nyházi. (N.S.). 2005. Vol. 21. № 1. P. 55–61.
11. Patel J., Cho N.E., Srivastava H.M. Certain subclasses of multivalent functions associated with a family of linear operators // Mathematical and Computer Modelling. 2006. Vol. 43. № 3–4. P. 320–338.

12. Raina R.K., Srivastava H.M. Inclusion and neighborhood properties of some analytic and multivalent functions // JIPAM. J. Inequal. Pure Appl. Math. 2006. Vol. 7. № 1. Paper № 5. 6 p.
13. Ruscheweyh S. New criteria for univalent functions // Proc. Amer. Math. Soc. 1975. Vol. 49. № 1. P. 109–115.
14. Ruscheweyh S. Neighborhoods of univalent functions // Proc. Amer. Math. Soc. 1981. Vol. 81. № 4. P. 521–527.
15. Silverman H. Univalent functions with negative coefficents // Proc. Amer. Math. Soc. 1975. Vol. 51. № 1. P. 109–116.

Поступила в редакцию 29.04.2015

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